نظریه بازیها
Game Theory

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Non-zero sum games

Material

- Dynamic Non-cooperative Game Theory: Second Edition
  - Chapter 3.6
  - Chapter4: Sections 4:1–4:3.

Non-zero sum games

- Zero sum games
- Non-zero sum games
  - N-player games
  - Bimatrix formulation
  - Nash equilibrium in mixed strategies
  - Completely mixed NE
  - Computing mixed NE
  - **Stackelberg games**
  - **Stackelberg vs. bilevel optimization**

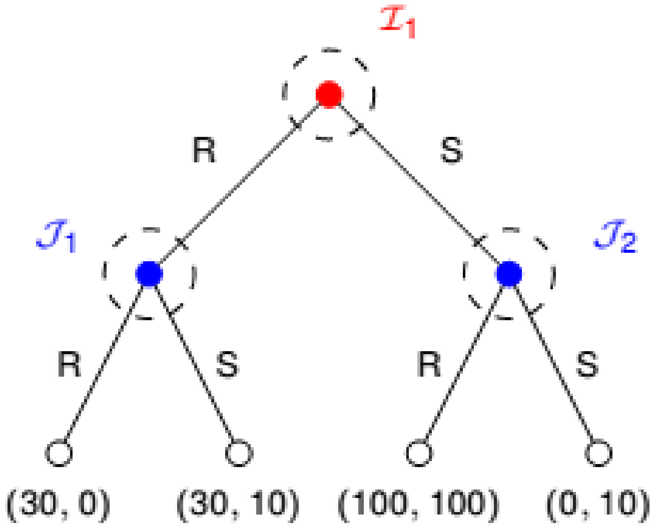
Stackelberg Games

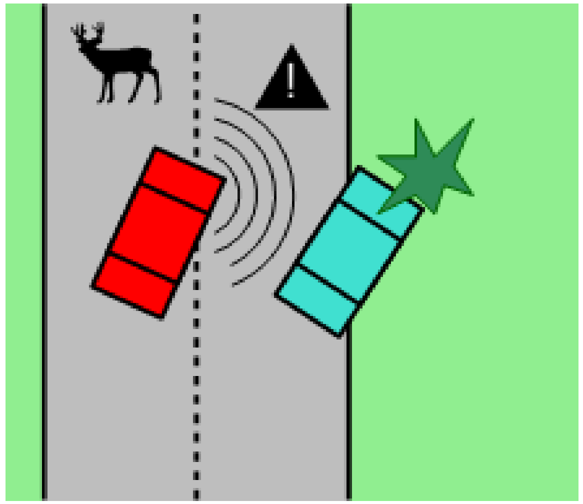
- **Consider a class of 2-player non-zero-sum games with**
  - sequential decision
  - single stage
  - perfect information
- **Asymmetric roles**
  - Leader: Player 1, chooses first and announces its decision
  - Follower: Player 2, chooses second, and knows Player 1 choice.

- **We define these games Stackelberg games, and their Nash equilibrium Stackelberg solution.**

Example: self driving car

- when car 1 announces its decision to car 2.





- Nash equilibria (no regret strategy)
  - (remain,remain), (swerve,swerve)
- Stackelberg equilibrium
  - (swerve,swerve)

Rational reaction

- pure-strategy space of the leader (Player 1): $\Gamma^{(1)}$
- pure-strategy space of the follower (Player 2): $\Gamma^{(2)}$

Rational Reaction

For each pure strategy $\gamma^{(1)} \in \Gamma^{(1)}$ of the leader, we define the rational reaction set $R(\gamma^{(1)}) \subseteq \Gamma^{(2)}$ as

$$R(\gamma^{(1)}) = \{\gamma^{(2)} \in \Gamma^{(2)} \mid J_2(\gamma^{(1)}, \gamma^{(2)}) \leq J_2(\gamma^{(1)}, \xi) \quad \forall \xi \in \Gamma^{(2)}\}$$

$R(\gamma^{(1)})$ is a **set**: there might be multiple responses to $\gamma^{(1)}$ which are equivalent for Player 2

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix}$$

$$R(\gamma_1^{(1)}) = \{\gamma_1^{(2)}\}$$
$$R(\gamma_2^{(1)}) = \{\gamma_1^{(2)}, \gamma_2^{(2)}\}$$

Stackelberg equilibrium strategy

Stackelber equilibrium

A strategy $\gamma^{(1)} \in \Gamma^{(1)}$ is called a Stackelberg equilibrium strategy for the leader (Player 1) if

$$\gamma^{(1)} \in S := \arg \min_{\gamma^{(1)} \in \Gamma^{(1)}} \max_{\gamma^{(2)} \in R(\gamma^{(1)})} J_1(\gamma^{(1)}, \gamma^{(2)})$$

The resulting outcome J_1^* is the Stackelberg cost of the leader.

- The Stackelberg equilibrium strategy is **not necessarily unique.**
- J_1^* is a conservative bound (**worst case**) on J_1.

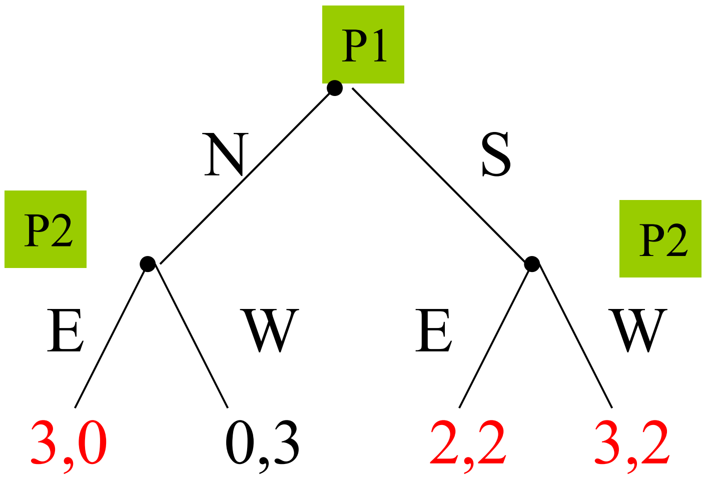
Example:

Rational reaction sets:

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix}$$

$$R(N) = \{E\}$$
$$R(S) = \{E, W\}$$

Worst rational reaction (for the leader):



$$\max_{\gamma^{(2)} \in R(N)} J_1(N, \gamma^{(2)}) = 3$$

$$\max_{\gamma^{(2)} \in R(S)} J_1(S, \gamma^{(2)}) = 3$$

Therefore $\quad \boldsymbol{S} = \{N, S\}$

Compact pure-strategy space

- **Assume compact pure-strategy space for both Players.**
  Example :

- $\{0,1\}$
- [a,b]
- $[0,1]^n \subset R^n$

but not

- N, R^n

- Stackelberg games of this kind can be interpreted as constrained optimization problems

$$Player 1 : x \in X \qquad Player 2 : y \in Y$$

Bilevel optimization problem

Bilevel optimization problem

A bilevel optimization program has the structure

$$\min_{x,y} \quad J_1(x,y)$$

$$\text{subject to} \quad x \in X$$

$$y = r(x)$$

Where

$$r(x) = \arg\min_{y \in Y(x)} J_2(x,y)$$

- Stackelberg game in an optimization framework, assuming the reaction set *R* is a singleton: $R = \{r(x)\}$.

Stackelberg vs. bilevel optimization

We have considered Stackelberg games

- one stage, sequential play
- two players (leader and follower), non-zero-sum
- compact action set
- no mixed strategies (perfect information)

The solution of Stackelberg games is closely connected to solving bilevel optimization problems

- non convex
- in general hard to solve!

An application in control of chemical processes

Problem

Determine the chemical composition of a complex mixture under **chemical equilibrium conditions**, at a given temperature and pressure.

Example: Burning rocket fuel(at 3500°, 50 atm)N_2H_4(Hydrazine)O_2

$H, H_2, H_2O, N, N_2, NH, NO, O, O_2, OH$

In what ratio?

Chemical equilibrium and Free energy

Chemical equilibrium model

A mixture of chemical species held at a constant temperature and pressure reaches its chemical equilibrium state concurrently with reduction of the free energy of the mixture to a minimum.

It follows from the second law of thermodynamics.

Parameters of the problem:

- Mixture of *m* chemical elements (N, H, …)
- *n* possible compounds (N_2H_4, H, O_2, …)
- c_{ij} is the number of atoms of element *i* in compound *j*

$$C = [c_{ij}] \in \mathrm{N}^{m \times n}$$

Free energy

- For example :

$$
C^\top = \begin{array}{c} \\ \text{H} \\ \text{H}_2 \\ \text{H}_2\text{O} \\ \text{N} \\ \text{N}_2 \\ \text{NH} \\ \text{NO} \\ \text{O} \\ \text{O}_2 \\ \text{OH} \end{array}
\begin{array}{c} \begin{array}{ccc} \text{H} & \text{N} & \text{O} \end{array} \\
\left[\begin{array}{ccc} 1 & & \\ 2 & & \\ 2 & & 1 \\ & 1 & \\ & 2 & \\ 1 & 1 & \\ & 1 & 1 \\ & & 1 \\ & & 2 \\ 1 & & 1 \end{array}\right] \end{array}
$$

Free energy

- $y_j, j = 1, \ldots, n$: moles of the compound j in the mixture
- $\bar{y} = \sum_{j=1}^{n} y_j$: total number of moles.

Gibbs free energy

$$E(x, y) = \sum_{j=1}^{n} y_j [k_j(x) + \log(\frac{y_j}{\bar{y}})]$$

Where

$$k_j(x) = \left(\frac{\mu_j(x)}{Rx}\right) + \log P$$

Where x is the process temperature, and parameters are known.

Chemical equilibrium

The composition of the mixture at chemical equilibrium is the solution of **the free energy minimization**

$$\min_{y_j, j=1,\ldots,n} E(x, y) = \sum_{j=1}^{n} y_j [\mathrm{k}_j (x) + \log(\frac{y_j}{\bar{y}})]$$

subject to mass balance constraints

$$\sum_{j=1}^{n} c_{ij} y_j = b_i, \qquad y_j \geq 0,$$

Where b_i is the amount of element *i* in the mixture.

Chemical equilibrium (example)

Burning rocket fuel (at 3500°, 50 atm): $\frac{1}{2}N_2H_4 + \frac{1}{2}O_2$

$$C^\top = \begin{array}{c} \\ H \\ H_2 \\ H_2O \\ N \\ N_2 \\ NH \\ NO \\ O \\ O_2 \\ OH \end{array} \begin{array}{c} \begin{array}{ccc} H & N & O \end{array} \\ \left[\begin{array}{ccc} 1 & & \\ 2 & & \\ 2 & & 1 \\ & 1 & \\ & 2 & \\ 1 & 1 & \\ & 1 & 1 \\ & & 1 \\ & & 2 \\ 1 & & 1 \end{array}\right] \end{array}$$

Amount of elements in the mixture:

- H: $b_1 = 2$
- N: $b_2 = 1$
- O: $b_3 = 1$

Mass balance constraints: $y \in R^{10}_{\geq 0}$

$$Cy = b$$

Chemical equilibrium (example)

$$\min_{y_j, j=1,\ldots,n} \quad E(y, x = T)$$

$$\text{subject} \quad Cy = b$$

Global minimization

The global minimum is the chemical equilibrium of the mixture.

→Sometimes hard to compute – although Nature does it really well!

Rocket fuel example:

	H	H_2	H_2O	N	N_2	NH	NO	O	O_2	OH
$y* =$	0.0407	0.1477	0.7831	0.0014	0.4853	0.0007	0.0274	0.0180	0.0373	0.0969

$$Cy = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

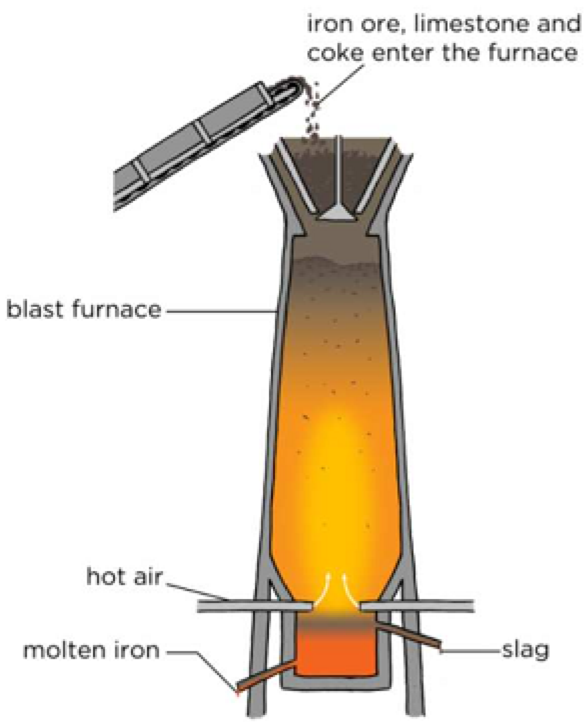
Chemical equilibrium and Stackelberg

$$\min_{y_j, j=1,\dots,n} \quad E(y, x = T)$$

$$\text{subject} \quad Cy = b$$

- Stackelberg game where
  - "Nature" is the follower, with
  - cost function $J_2(x,y) = E(x,y)$(Gibbs free energy)
  - Actions $y \in R^n_{\geq 0}$ (element mixture)
  - "decides" the chemical equilibrium y, given the temperature x
- We design the leader, a temperature controller with
  - our desired cost function $J_1(x,y)$ of temperature and mixture
  - Action x(temperature)
  - decides the temperature and allows the system to react

Iron Furnace



In Iron furnace, row materials are continuously added

- Oxygen
- Iron oxid
- Carbon
- Water

And the temperature *x* is controlled

Output:

CO CO_2 H_2 O_2 H_2O FeO Fe C

Cost :

Cost (x) –price (y_{Fe})

Iron Furnace

Stackelberg game

Leader: Controller – action space $x \in [T_{min}, T_{max}]$

Follower: Nature – action space $y \in Y = \{y \in R^{8}_{\geq 0} \mid Cy = b\}$

- $b = Cy^{in}$, where y^{in} is the input mixture of compounds

Bilevel optimization problem

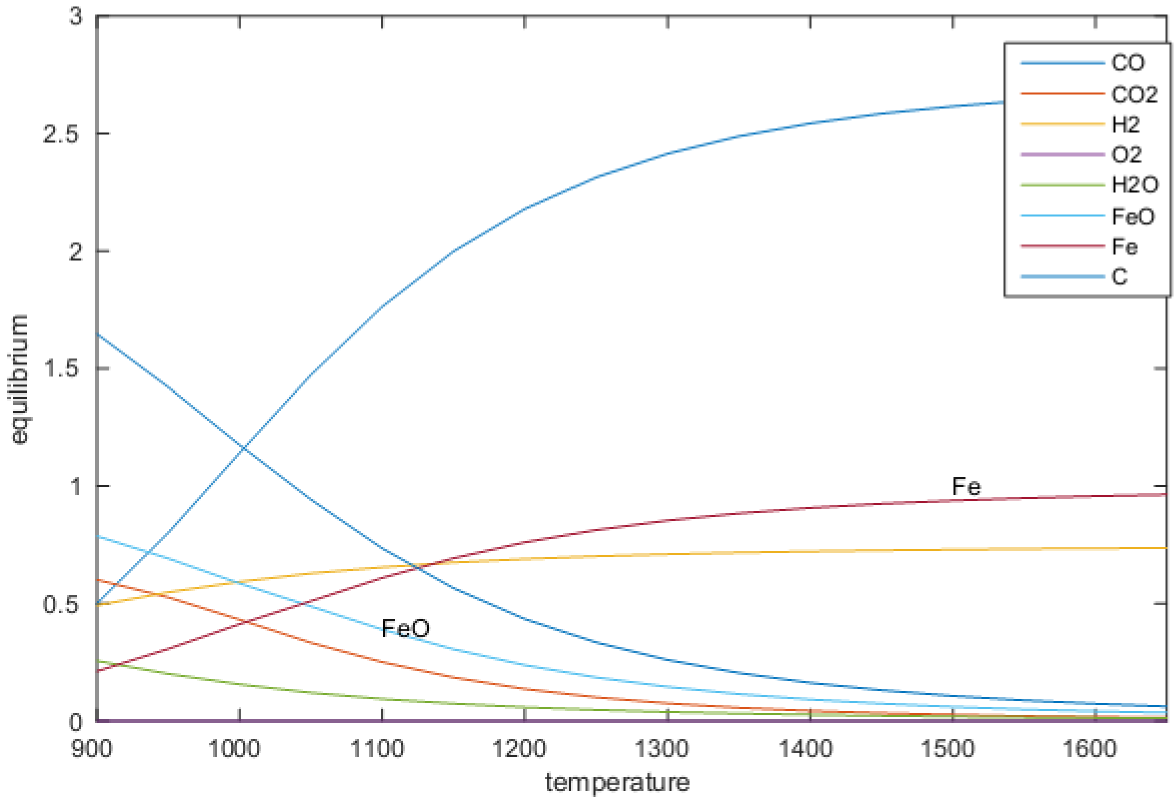
$$\min_{x,y} \quad J_1(x,y) = a_1 x + a_2 x^2 - \gamma y_{Fe}$$

$$\text{subject to} \quad x \in [T_{min}, T_{max}]$$

$$y = R(x)$$

Where $R(x) = \arg\min_{y \in Y(x)} E(x,y)$

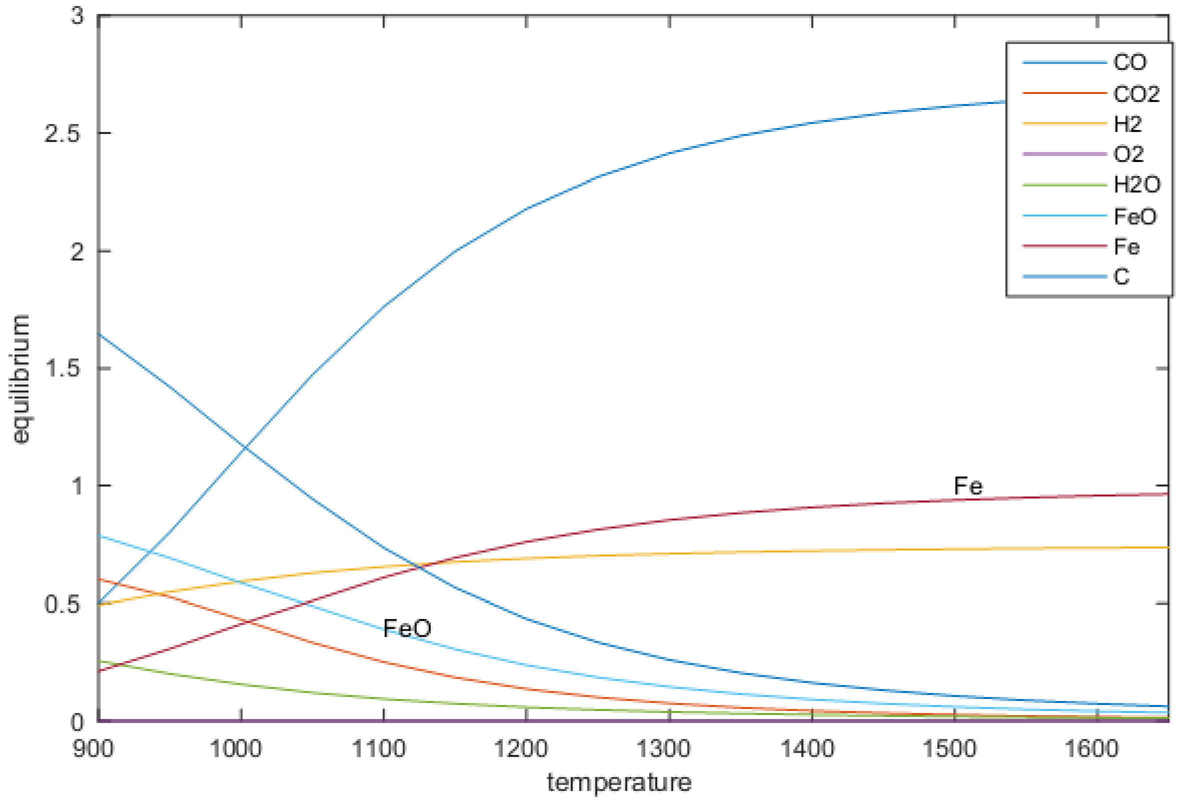
Equilibria



- For each temperature x:

$$y^* = \arg\min_{y \in Y(x)} E(x, y) \text{ subject to } \mathrm{C}y = \mathrm{b}, y \geq 0 \quad (\mathrm{b}_{Fe} = 1)$$

Equilibria

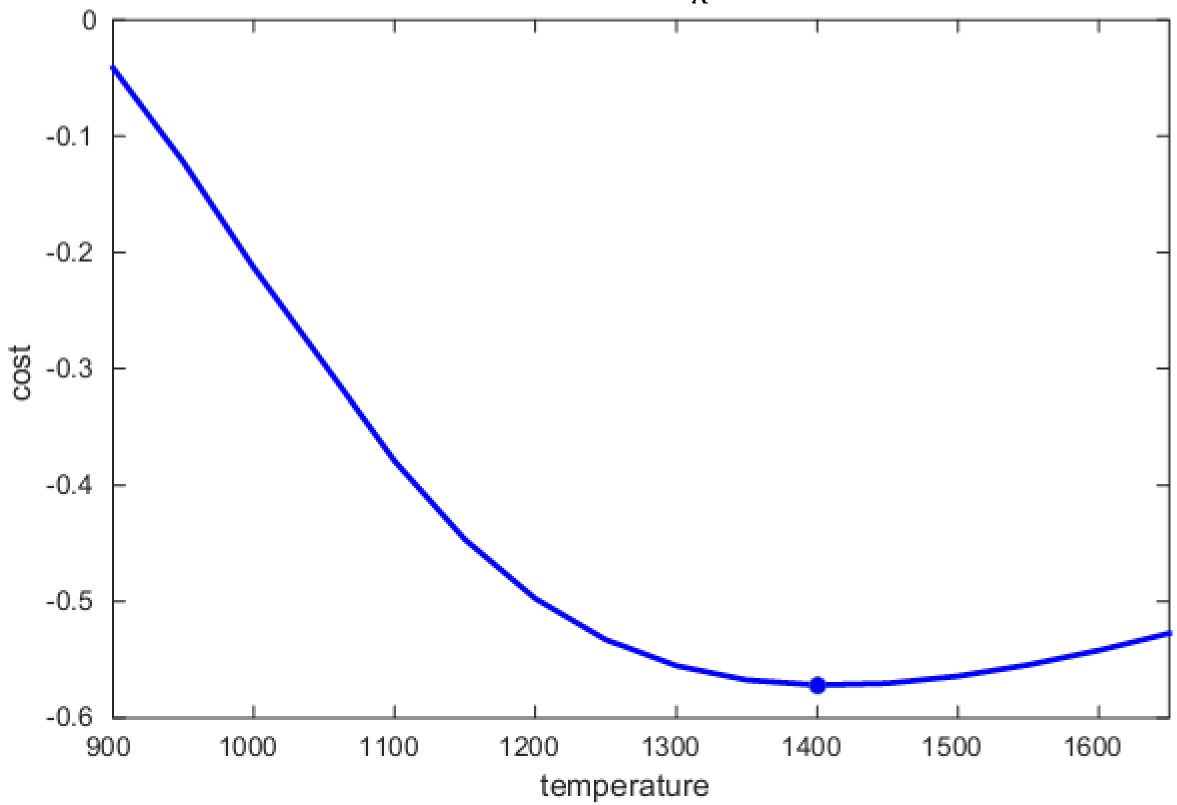


- Example : x=1400°

$$R(x) = [2.54 \quad 0.04 \quad 0.72 \quad 0.00 \quad 0.03 \quad 0.09 \quad 0.91 \quad 0.16]^T$$

Cost function:

- Plot $J_1(x, y^*) = a_1 x + a_2 x^2 - \gamma y_{Fe}$
- Find the temperature $x^* = \arg\min_x J_1(x, y^*)$



Applications of Stackelberg games

- Other applications:
- Traffic control
- Chemical Process Synthesis
- Market
- etc.