# نظریه بازیها
# Game Theory

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# Stochastic games

Material

- Mertens, J. F. & Neyman, A. (1981). "Stochastic Games". International Journal of Game Theory. 10 (2): 53–66.
- Neyman, A. & Sorin, S. (2003). "Stochastic Games and Applications". Dordrecht: Kluwer Academic Press.
- Shapley, L. S. (1953). "Stochastic games". PNAS. 39 (10): 1095–1100.

# Stochastic games

- Zero sum games
- Non-zero sum games
- Infinite Games
- Infinite Dynamic Games
- **Stochastic games**
  - **Markov Games**
  - **Evolutionary Games**

# Stochastic Games

- A stochastic game is a collection of normal-form games that the agents play repeatedly
- The particular game played at any time depends probabilistically on
  - the previous game played
  - the actions of the agents in that game

# Markov Games

- A **stochastic** (or **Markov**) game includes the following:
  - a finite set Q of states (games),
  - a set $N = \{1, 2, ..., n\}$ of agents,
  - For each agent *i*, a finite set $A_i$ of possible actions
  - A transition probability function $P : Q \times A_1 \times \ldots \times A_n \times Q \to [0, 1]$
    $P(q, a_1, \ldots, a_n, q') =$ probability of transitioning to state $q'$ if the action profile $(a_1, \ldots, a_n)$ is used in state q
  - For each agent *i*, a real-valued payoff function
    $r_i : Q \times A_1 \times \ldots \times A_n \to R$
- This definition makes the inessential but simplifying assumption that each agent's strategy space is the same in all games
  - So the games differ only in their payoff functions

# Histories and Rewards

- Before, a history was just a sequence of actions
- But now we have action profiles rather than individual actions, and each profile has several possible outcomes
- Thus a history is a sequence $h_t = (q^0, a^0, q^1, a^1 \ldots, q^{t-1}, a^{t-1}, q^t)$, where t is the number of stages
- The two most common methods to aggregate payoffs into an overall payoff are **average reward** and **future discounted reward**
- Agent i's **average reward** is

$$\lim_{k \to \infty} \sum_{j=1}^{k} r_i^{(j)} / k$$

# Histories and Rewards

- Agent i's **future discounted reward** is the discounted sum of the payoffs, i.e., $\sum_{j=1}^{\infty} \beta^j r_i^{(j)}$
- Where $\beta$ $(with\ 0 \le \beta \le 1)$ is a constant called the discount factor
- Two ways to interpret the discount factor:

1. The agent cares more about the preset than the future
2. The agent cares about the future, but the game ends at any round with probability $1-\beta$

# Histories and Rewards

- Stochastic games generalize both Markov decision processes ( MDPs) and repeated games
- An MDP is a stochastic game with only 1 player
- A repeated game is a stochastic game with only 1 state
  - Iterated Prisoner's Dilemma, Roshambo (Rock, Paper, Scissors) ,
  - Iterated Battle of the Sexes,

# Strategies

- For agent i, a **deterministic strategy** specifies a choice of action for i at every stage of every possible history
- A mixed strategy is a probability distribution over deterministic strategies
- Several restricted classes of strategies:
  - As in extensive-form games, a **behavioral strategy** is a mixed strategy in which the mixing take place at each history independently
  - A **Markov strategy** is a behavioral strategy such that for each time t, the distribution over actions depends only on the current state
    - But the distribution may be different at time t than at time $t' \neq t$
  - A **stationary strategy** is a Markov strategy in which the distribution over actions depends only on the current state (not on the time t)

# Equilibria

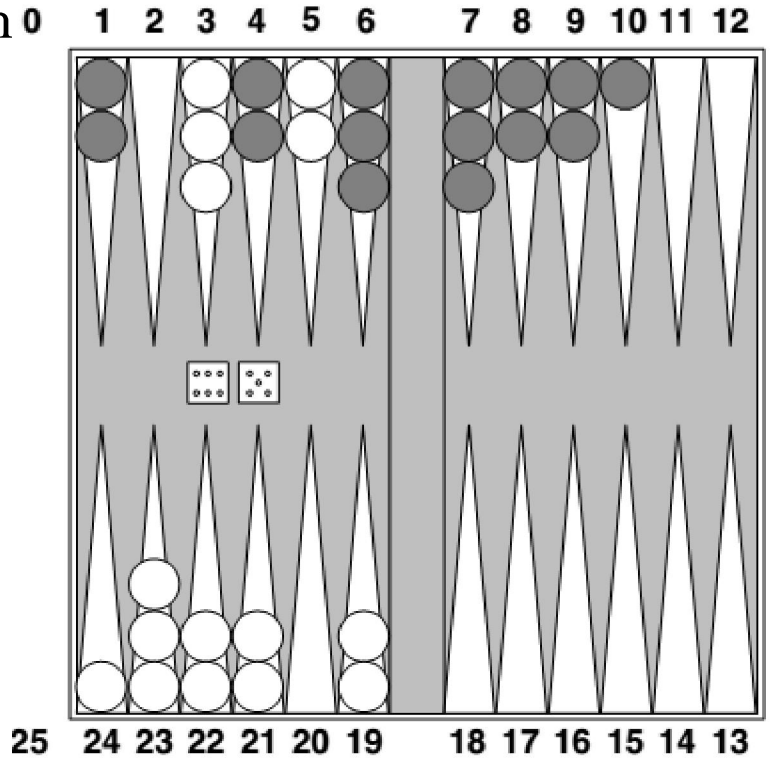
- First consider the (easier) discounted-reward case
- A strategy profile is a Markov-perfect equilibrium (MPE) if
  - it consists of only Markov strategies
  - it is a Nash equilibrium regardless of the starting state
- **Theorem**: Every n-player, discounted-reward stochastic game has a MPE
- The role of Markov-perfect equilibria is similar to role of subgame -perfect equilibria in perfect-information games

# Equilibria

- Now consider the average-reward case
- A stochastic game is **irreducible** if every game can be reached with positive probability regardless of the strategy adopted
- **Theorem**: Every 2-player, average reward, irreducible stochastic game has a Nash equilibrium
- A payoff profile is feasible if it is a convex combination of the outcomes in a game, where the coefficients are rational numbers
- There's a folk theorem similar to the one for repeated games:
- If $(p_1,p_2)$ is a feasible pair of payoffs such that each $p_i$ is at least as big as agent i's minimax value, then $(p_1,p_2)$ can be achieved in equilibrium through the use of enforcement

# Two-Player Zero-Sum Stochastic Games

- For two-player zero-sum stochastic games
- The situation is similar to what happened in repeated games
- The only feasible pair of payoffs is the minimax payoffs
- One example of a two-player zero-sum stochastic game is **Backgammon**
  - Two agents who take turns
  - Before his/her move, an agent must roll the dice
  - The set of available moves depends on the results of the dice roll



# Two-Player Zero-Sum Stochastic Games

- Mapping Backgammon into a Markov game is straightforward, but slightly awkward
- Basic idea is to give each move a stochastic outcome, by combining it with the dice roll that comes after it
- Every state is a pair:
- (current board, current dice configuration)
- Initial set of states = {initial board} ×

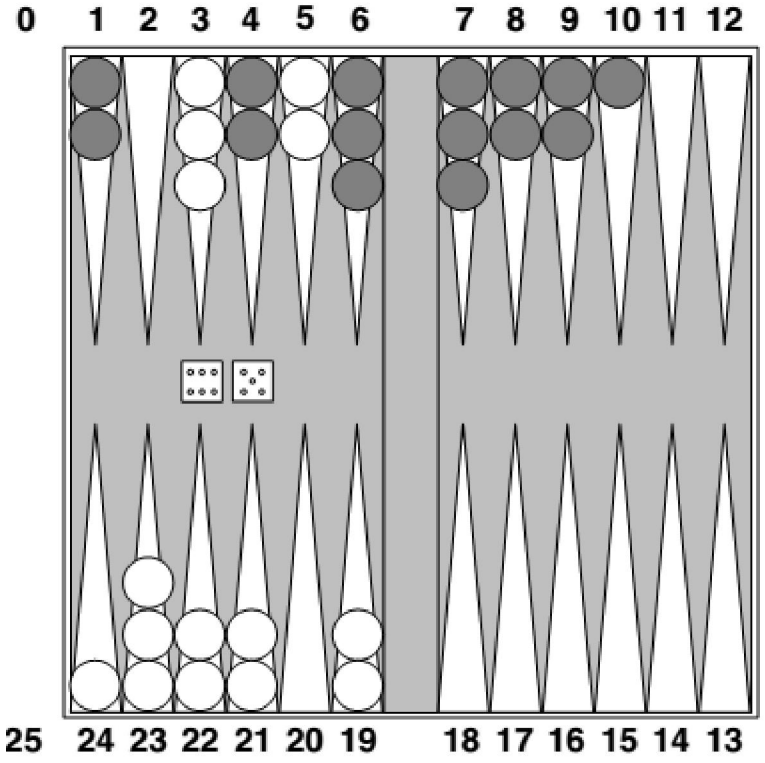
{all possible results of agent 1's first dice roll}

- Set of possible states after agent 1's move =

{the board produced by agent 1's move}

× {all possible results of agent 2's dice roll}

- Vice versa for agent 2's move



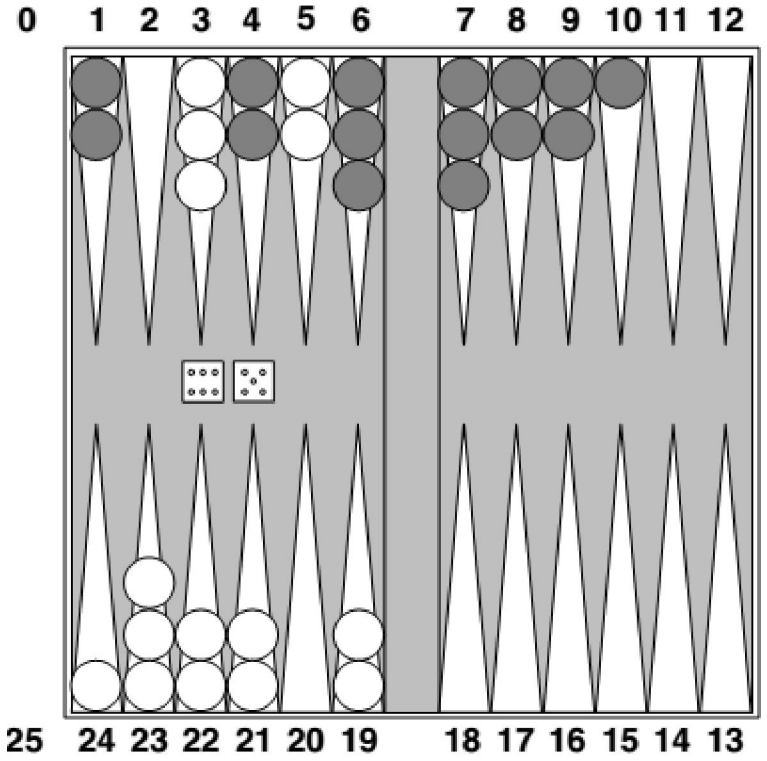
# Two-Player Zero-Sum Stochastic Games

- We can extend the minimax algorithm to deal with this
- But it's easier if we don't try to combine the moves and the dice rolls
- Just keep them separate
- Dice rolls increase branching factor
- 21 possible rolls with 2 dice
- Given the dice roll, ≈20 legal moves

on average

For some dice roles, can be much higher

- depth 4 = $20\times(21\times20)^3\approx1.2\times10^9$



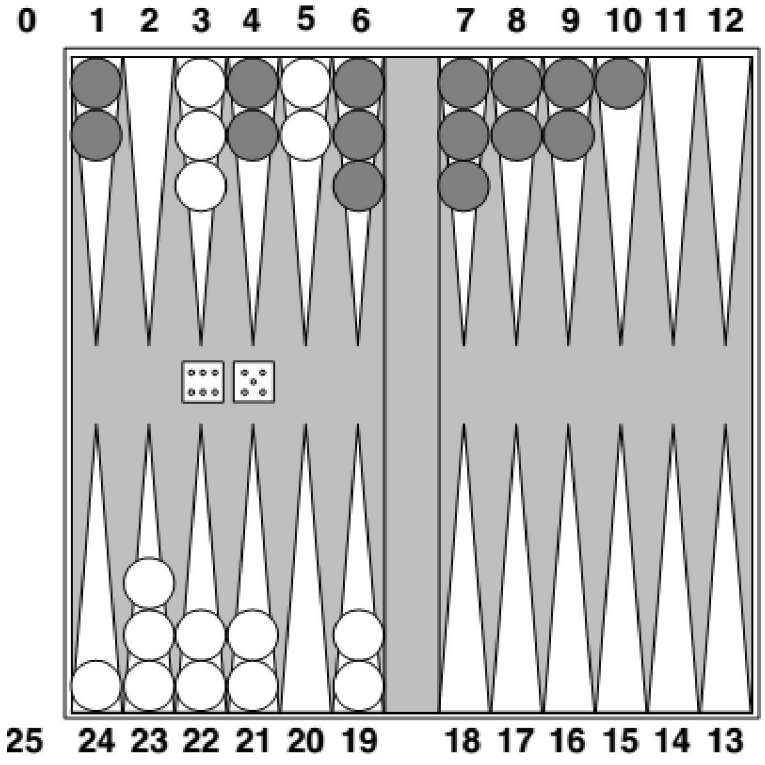
# Two-Player Zero-Sum Stochastic Games

- As depth increases, probability of reaching a given node shrinks
- TDGammon uses depth-2 search+very good evaluation function
- ≈world-champion level
- The evaluation function was

created automatically using

a machine learning technique

called Temporal Difference learning



# Stochastic games

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  - Markov Games
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# Evolutionary Simulations

- An evolutionary simulation is a stochastic game whose structure is intended to model certain aspects of evolutionary environments
  - At each stage(or generation) there is a large set (e.g., hundreds) of agents
- Different agents may use different strategies
  - A strategy s is represented by the set of all agents that use strategy s
  - Over time, the number of agents using s may grow or shrink depending on how well s performs
- s's reproductive success is the fraction of agents using sat the end of the simulation,
- i.e., (number of agents using s)/(total number of agents)

# Replicator Dynamics

- At each stage, some set of agents (maybe all of them, maybe just a few) is selected to perform actions at that stage
  - Each agent receives a fitness value: a stochastic function of the action profile
- Depending on the agents' fitness values, some of them may be removed and replaced with agents that use other strategies
  - Typically an agent with higher fitness is likely to see its numbers grow
  - The details depend on the **reproduction dynamics**
    - The mechanism for selecting which agents will be removed, which agents will reproduce, and how many progeny they'll have

# Replicator Dynamics

- **Replicator dynamics** works as follows:
- $p_i^{new} = p_i^{curr} r_i / R$,

where

- $p_i^{new}$ is the proportion of agents of type *i* in the next stage
- $p_i^{cur}$ is the proportion of agents of type *i* in the current stage
- $r_i$ = average payoff received by agents of type *i* in the current stage
- $R_i$ = average payoff received by all agents in the current stage
- Under the replicator dynamics, an agent's numbers grow (or shrink) proportionately to how much better it does than the average
- Probably the most popular reproduction dynamics
  - e.g., does well at reflecting growth of animal populations

# Replicator Dynamics

- **Imitation dynamics** (or tournament selection) works as follows:
  - Randomly choose 2 agents from the population, and compare their payoffs
    - The one with the higher payoff reproduces into the next generation
  - Do this n times, where n is the total population size
- Under the imitation dynamics, an agent's numbers grow if it does better than the average
  - But unlike replicator dynamics, the amount of growth doesn't depend on how much better than the average
- Thought to be a good model of the spread of behaviors in a culture

# Example: A Simple Lottery Game

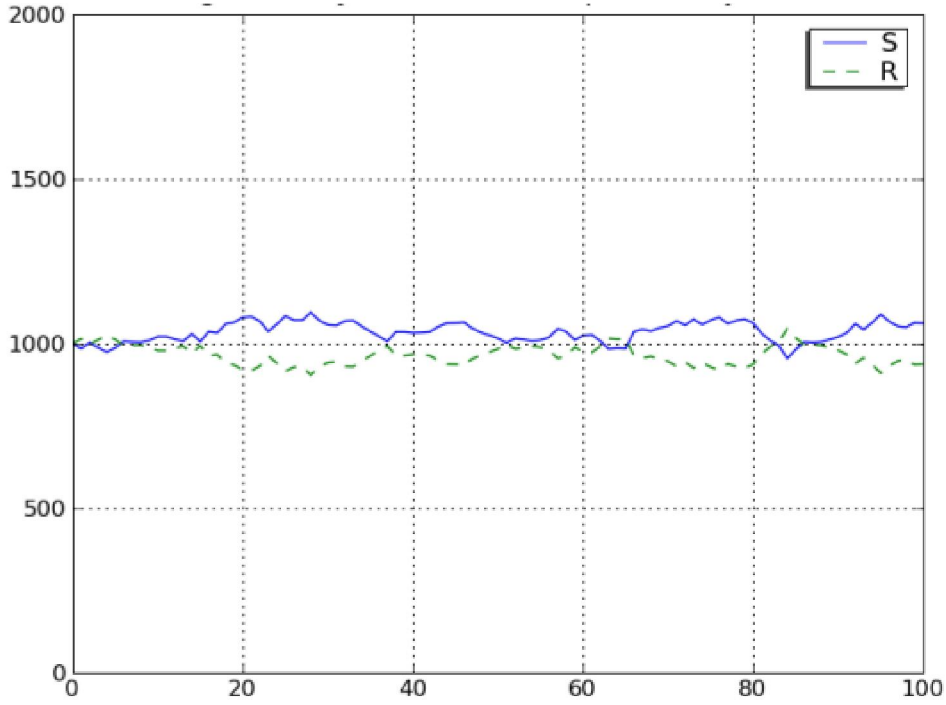
- A repeated lottery game
- At each stage, agents make choices between two lotteries
  - "Safe" lottery: guaranteed reward of 4
  - "Risky" lottery: [0, 0.5; 8, 0.5],
    - i.e., probability ½of 0, and probability ½of 8
- Let's just look at stationary strategies
- Two pure strategies:
  - S: always choose the "safe" lottery
  - R: always choose "risky" lottery
- Many mixed strategies, one for every p in [0,1]
- $R_p$: probability p of choosing the "risky" lottery, and probability 1–p of choosing the "safe" lottery

# Lottery Game with Replicator Dynamics

- At each stage, each strategy's average payoff is 4
  - Thus on average, each strategy's population size should stay roughly constant
- Verified by simulation

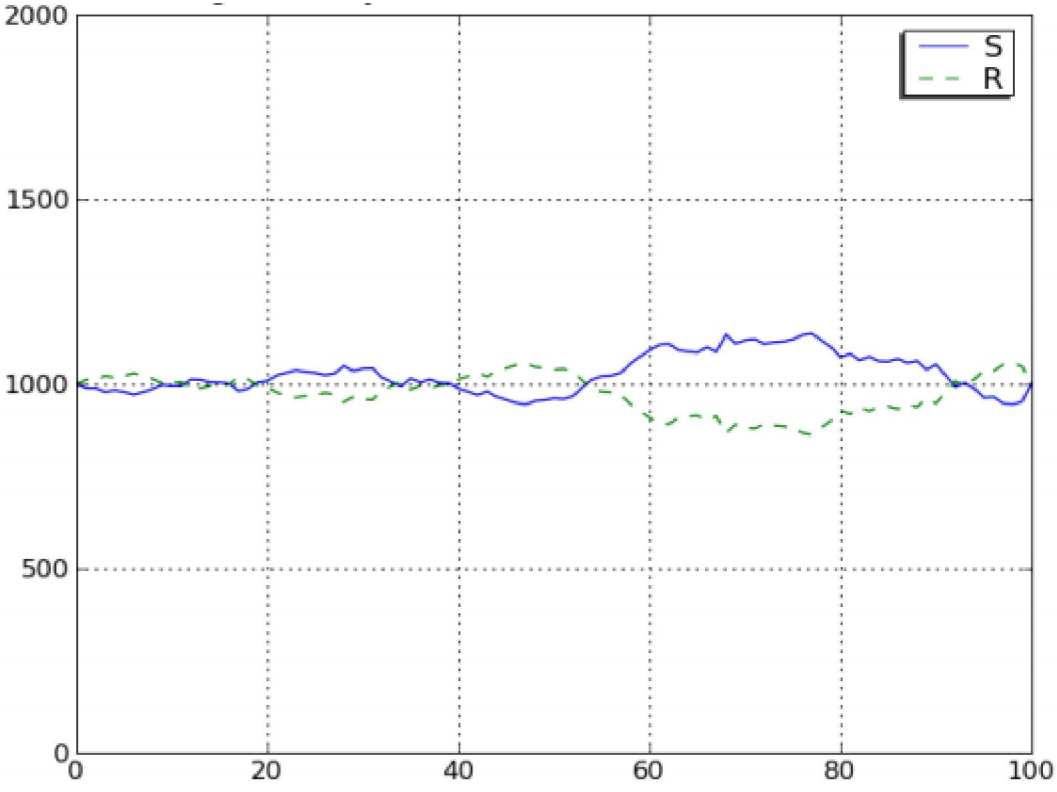
for S and R

- Would get similar behavior
- with any of the $R_p$ strategies



# Lottery Game with Imitation Dynamics

- Pick any two agents, and let sand tbe their strategies
- Regardless of what s and t are, each agent has equal probability of getting a higher payoff than the other
- Again, each strategy's population size should stay roughly constant
- Verified by simulation for S and R
- Would get similar behavior with any of the $R_p$ strategies

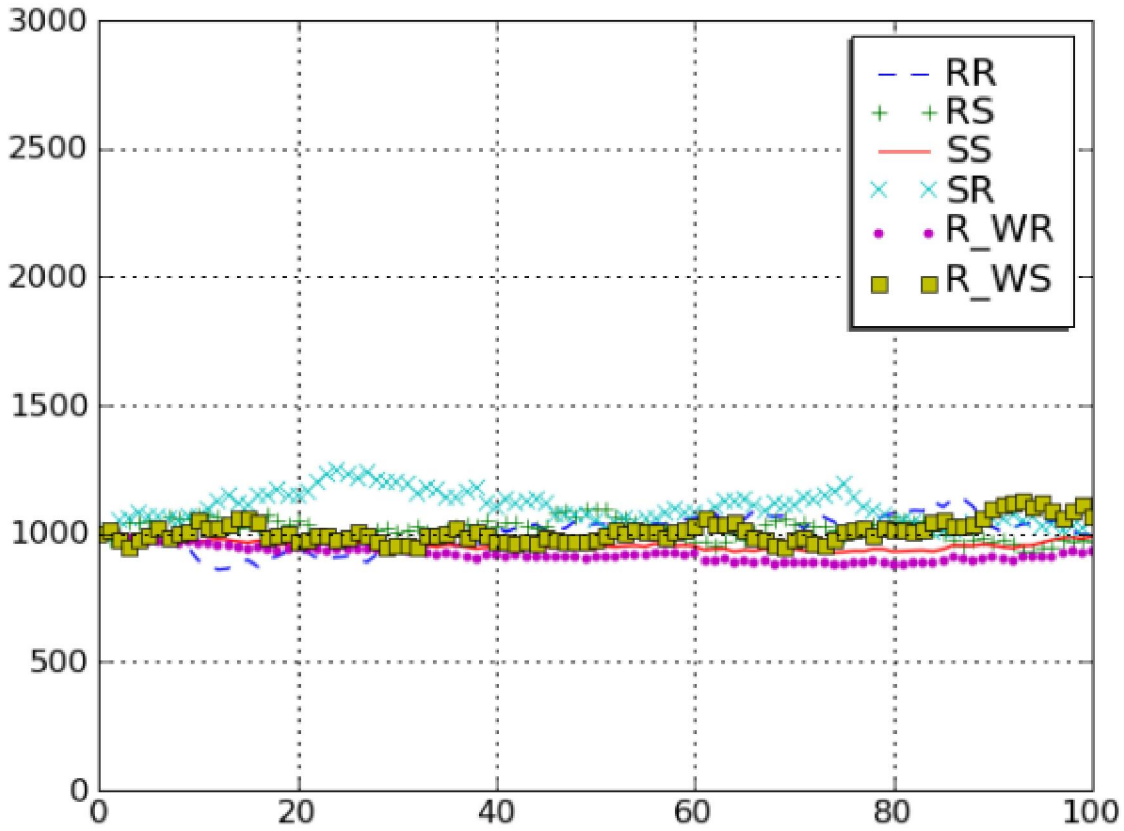


# Double Lottery Game

- Now, suppose that at each stage, agents make two rounds of lottery choices
  1. Choose between the safe or risky lottery, get a reward
  2. Choose between the safe or risky lottery again, get another payoff
- This time, there are 6 stationary pure strategies
  - SS: choose "safe" both times
  - RR: choose "risky" both times
  - SR: choose "safe" in first round, "risky" in second round
  - RS: choose "risky" in first round, "safe" in second round
  - R-WR: choose "risky" in first round
    - If it wins (i.e., reward is 8), then choose "risky" again in second round
    - Otherwise choose "safe" in second round
  - R-WS: choose "risky" in first round
    - If it wins (i.e., reward is 8), then choose "safe" in second round
    - Otherwise choose "risky" in second round

# Double Lottery Game, Replicator Dynamics

- At each stage, each strategy's average payoff is 8
  - Thus on average, each strategy's population size should stay roughly constant
- Verified by simulation for all 6 strategies



# Double Lottery Game, Imitation Dynamics

- Pick any two agents a and b, and let choose actions
  - Reproduce the agent (hence its strategy) that wins (i.e., higher reward)
  - If they get the same reward, choose one of them at random Verified by simulation
- We need to look at each strategy's distribution of payoffs:

| *R-WS* | | | *R-WR* | | | *SR* | | *RS* | | *SS* | *RR* | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **12** | **8** | **0** | **16** | **8** | **4** | **12** | **4** | **12** | **4** | **8** | **16** | **8** | **0** |
| .5 | .25 | .25 | .25 | .25 | .5 | .5 | .5 | .5 | .5 | 1 | .25 | .5 | .25 |

- Suppose a uses SS and b uses SR
  - P(SR gets 12 and SS gets 8) = (0.5)(1.0) = 0.5 => SRwins
  - P(SRgets 4 and SSgets 8) = (0.5)(1.0) = 0.5 => SSwins
  - Thus a and b are equally likely to reproduce
- Same is true for any two of {SS, SR, RS, RR}

# Double Lottery Game, Imitation Dynamics

| R-WS | | | R-WR | | | SR | | RS | | SS | RR | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 8 | 0 | 16 | 8 | 4 | 12 | 4 | 12 | 4 | 8 | 16 | 8 | 0 |
| .5 | .25 | .25 | .25 | .25 | .5 | .5 | .5 | .5 | .5 | 1 | .25 | .5 | .25 |

- Suppose a uses R-WS and b uses SS
  - Even though they have the same expected reward, R-WS is likely to get a slightly higher reward than SS:
    - P(R-WS gets 12 and SS gets 8) = (0.5)(1.0) = 0.5 => R-WS wins
    - P(R-WS gets 8 and SS gets 8) = (0.25)(1.0) = 0.25 => tie
    - P(R-WS gets 0 and SS gets 8) = (0.25)(1.0) = 0.25 => SS wins
  - Thus a reproduces with probability 0.625,

  and b reproduces with probability 0.375
- Similarly, a is more likely to reproduce than b

if a uses R-WS and b uses any of {SS, RR, R-WR}

# Double Lottery Game, Imitation Dynamics

- If we start with equal numbers of all 6 strategies, S-WRwill increase until SS, RR, and R-WR become extinct
  - The population should stabilize with a high proportion of S-WR, and low proportions of SR and RS
  - Verified by simulation