نظریه بازیها
Game Theory

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Bayesian Games

Material

- Fudenberge D., Birole J., Game Theory, MIT Press, Cambridge, Massachusetts, 1991,Sections 6.1-6.5.

Bayesian Games

- Zero sum games
- Non-zero sum games
- Infinite Games
- Infinite Dynamic Games
- Stochastic games
- **Bayesian games**
  - Incomplete information.
  - Bayes rule and Bayesian inference.
  - **Bayesian Nash Equilibria**.
  - **Auctions.**
  - **Dynamic Games of Incomplete Information**

Bayesian Games (Example)

- Suppose that the payoffs are given by the table

	L	R
X	θ, γ	1,2
Y	$-1, \gamma$	$\theta, 0$

Where $\theta \in \{0, 2\}$ is known by Player 1, $\gamma \in \{1, 3\}$ is known by Player 2, and all pairs of (θ, γ) have probability of 1/4.

- Formally, the Bayesian game is defined as
- $N = \{1, 2\}$.

Bayesian Games (Example)

- $T_1 = \{0,2\}, T_2 = \{1,3\}$.
- $p(0,1) = p(0,3) = p(2,1) = p(2,3) = 1/4.$
- $A_1 = \{X, Y\}, A_2 = \{L, R\}$.
- u_1 and u_2 are defined by the table above

For computing a Bayesian Nash equilibrium s^* of this game. To do that, one needs to determine $s_1^*(0) \in \{X, Y\}, s_1^*(2) \in \{X, Y\}$ $, s_2^*(1) \in \{L, R\}$ and $s_2^*(3) \in \{L, R\}$. Four actions in total. First observe that when $\theta = 0$,action X strictly dominates action Y , i.e.,

$$u_1(X, a_2, \theta = 0, \gamma) > u_1(Y, a_2, \theta = 0, \gamma)$$

Bayesian Games (Example)

- for all actions $a_2 \in A_2$ and types $\gamma \in \{1,3\}$ of Player 2. Hence, it must be that $s_1^*(0) = X$
- Similarly, when $\gamma = 3$, action L strictly dominates action R, and hence $s_2^*(3) = L$
- Now consider the type $\theta = 2$ of Player 1. Since his payoff does not depend on γ observe that his payoff from X is $1+ p_L$, where p_L is the probability that Player 2 plays L. His payoff from Y is $2(1- p_L)- p_L$, which is equal to $2- 3p_L$. Hence, for $\theta = 2$, X is a best response if

$$1 + p_L \geq 2 - 3p_L \qquad \Rightarrow p_L \geq 1/4$$

Bayesian Games (Example)

- When $p_L \geq 1/4$, X is the only best response. Note however that type $\gamma = 3$ must play L, and the probability of that type is 1/2. Therefore, $p_L \geq 1/2 > 1/4$
- Since $s_1^*(2)$ is a best response for $\theta = 2$, it follows that

$$s_1^*(2) = X \qquad \gamma \in \{1,3\}$$

- Now consider $\gamma = 1$. Given s_1^*, Player 2 knows that Player 1 plays X(regardless of his type). Hence, the payoff of $\gamma = 1$ is $\gamma = 1$ when he plays L and 2 when he plays R. Therefore,

$$s_2^*(1) = R$$

- To check that s^* is indeed a Bayesian Nash equilibrium, one checks that each type plays a best response.

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Auctions

- A major application of Bayesian games is to **auctions.**
- This corresponds to a situation of incomplete information because the valuations of different potential buyers are unknown.
- We made the distinction between
  - **Private value auctions**: valuation of each agent is independent of others' valuations;
  - **Common value auctions:** the object has a potentially common value, and each individual's signal is imperfectly correlated with this common value

Types of Auctions

- Classification according to the rules for bidding
  - English
  - Dutch
  - First price sealed bid
  - Vickrey (Second-Price Auctions Sealed-Bid)
  - many others
- A possible problem is collusion(secret agreements for fraudulent purposes)
  - Groups of bidders who won't bid against each other, to keep the price low
  - Bidders who place phony (phantom) bids to raise the price (hence the auctioneer's profit)
- If there's collusion, the equilibrium analysis is no longer valid

English Auction

- The name comes from **oral auctions** in English-speaking countries
  - But I think this kind of auction was also used in **ancient Rome**
- **Commodities:**
  - antiques, artworks, cattle, horses, real estate, wholesale fruits and vegetables, old books, etc.
- **Typical rules:**
  - Auctioneer first solicits an opening bid from the group
  - Anyone who wants to bid should call out a new price at least x higher than the previous high bid (e.g., x= 1 Euro)
  - The bidding continues until all bidders but one have dropped out
  - The highest bidder gets the object being sold, for a price equal to the final bid
- Winner's profit = BV – price
- Everyone else's profit = 0

English Auction

- **Optimal strategy:**
  - participate until highest bid = your valuation of the commodity, then drop out
- **Equilibrium Outcome**
  - The highest bidder gets the object, at a price close to the second highest BV
- **Let n be the number of bidders**
  - The higher n is, the closer winning bid is to the highest BV.
  - If there is a large range of BVs, then the difference between the highest and 2nd-highest BVs may be large
    - Thus if there's wide disagreement about the item's value, the winner might be able to get it for much less than his/her BV

First-Price Sealed-Bid Auctions

- **Examples:**
  - construction contracts (lowest bidder)
  - real estate
  - art treasures
- **Typical rules:**
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to his/her own bid
  - Winner's profit = BV – price
  - Everyone else's profit = 0

First-Price Sealed-Bid Auctions

- Suppose that:
  - There are n bidders
  - Each bidder has a private valuation, v_i, which is private information
  - But a probability distribution for v_i is common knowledge
    - Let's say v_i is uniformly distributed over [0, 1]
  - Let B_i denote the bid of player i
  - Let u_i denote the profit of player i
- What is the Bayes-Nash equilibrium bidding strategy for the players?
  - Need to find the optimal bidding strategies
- First we'll look at the case where n= 2

First-Price Auctions for two bidders

- Simultaneously, each bidder i submits a bid $b_i \geq 0$. Then, the highest bidder wins the object and pays her bid.
- If they bid the same number, then the winner is determined by a coin toss.
- The value of the object for bidder i is v_i, which is privately known by bidder i. That is, v_i is the type of bidder i. Assume that v_1 and v_2 are "independently and identically distributed“ with uniform distribution over [0,1].
- This precisely means that knowing her own value v_i, bidder i believes that the other bidder's value v_j is distributed with uniform distribution over [0,1], and the type space of each player is [0,1].
- Recall that the beliefs of a player about the other player's types may depend on the player's own type. Independence assumes that it doesn't.

First-Price Auctions for two bidders

- Formally, the Bayesian game is as follows. Actions are b_i, coming from the action spaces $[0,\infty)$;types are v_i, coming from the type spaces $[0,1]$; beliefs are uniform distributions over $[0,1]$ for each type, and the utility functions are given by

$$u_i(b_1,b_2,v_1,v_2)=\begin{cases} v_i-b_i & \text{if } b_i>b_j \\ \dfrac{v_i-b_i}{2} & \text{if } b_i=b_j \\ 0 & \text{if } b_i<b_j \end{cases}$$

- In a Bayesian Nash equilibrium, each type v_i maximizes the expected payoff over b_i

$$E[u_i(b_1,b_2,v_1,v_2)\,|\,v_i]=(v_i-b_i)\Pr\{b_i>b_j(v_j)\}+\frac{v_i-b_i}{2}\Pr\{b_i=b_j(v_j)\}$$

First-Price Auctions for two bidders

- First, we consider a special equilibrium. The technique we will use here is a common technique in computing Bayesian Nash equilibria, and pay close attentions to the steps.

Symmetric, linear equilibrium

- Symmetric means that equilibrium action $b_i(v_i)$ of each type v_i is given by

$$b_i(v_i) = b(v_i)$$

- for some function b from type space to action space, where b is the same function for all players. Linear means that b is an affine function of v_i

$$b_i(v_i) = a + cv_i$$

First-Price Auctions for two bidders

To compute symmetric, linear equilibrium, one follows the following steps:

1)Assume a symmetric "linear" Bayesian Nash equilibria :

$$b_1^*(v_1) = a + cv_1$$

$$b_2^*(v_2) = a + cv_2$$

2)Compute best reply function of each type:

$$\mathrm{b}_i(v_i) = (a + v_i) / 2$$

3)Verify that best reply functions are affine:

$$\mathrm{b}_i(v_i) = \frac{a}{2} + \frac{v_i}{2}$$

First-Price Auctions for two bidders

4) Compute the constants a and c:

$$a + cv_i = \frac{a}{2} + \frac{v_i}{2} \Rightarrow a = \frac{a}{2} \,\&\, \mathrm{c} = \frac{1}{2}$$

$$\Rightarrow a = 0 \,\&\, \mathrm{c} = \frac{1}{2}$$

Note that we took an integral to compute the expected payoff and took a derivative to compute the best response.

First-Price Auctions for two bidders

We now compute a symmetric Bayesian Nash equilibrium without assuming that b is linear. Assume that b is strictly increasing and differentiable.

1)Assume a symmetric BNE (of the form):

$$b_1^*(v_1) = b(v_1)$$

$$b_2^*(v_2) = b(v_2)$$

2)Compute the (1st-order condition for) best reply of each type:

$$\mathrm{E}[\mathrm{u}_i(b_i, b_j^*, v_1, v_2) \mid \mathrm{v}_i] = (v_i - b_i)\Pr\{b_i > b(v_j)\}$$

$$= (v_i - b_i)\Pr\{v_j < b^{-1}(b_i)\}$$

$$= (v_i - b_i)b^{-1}(b_i)$$

First-Price Auctions for two bidders

where b^{-1} is the inverse of b. Here, the first equality holds because b is strictly increasing; the second equality is obtained by again using the fact that b is increasing, and the last equality is by the fact that v_j is uniformly distributed on [0,1]. The first-order condition is obtained by taking the partial derivative of the last expression with respect to b_i and setting it equal to zero. Then, the first-order condition is

$$-b^{-1}(b_i^*) + (v_i - b_i)\frac{db^{-1}}{db_i}\Big|_{b_i = b_i^*} = 0$$

3)Identify best reply with Bayesian Nash equilibrium action:

$$b_i^*(v_i) = b(v_i)$$

First-Price Auctions for two bidders

4) Substitute 3 in 2:

$$-v_i + (v_i - b(v_i))\frac{1}{b'(v_i)} = 0$$

5) Solve the differential equation (if possible):

Most of the time the differential equation does not have a closed-form solution. In that case, one suffices with analyzing the differential equation. Luckily, in this case the differential equation can be solved, easily. By simple algebra, we rewrite the differential equation as

$$b'(v_i)v_i + b(v_i) = v_i \Rightarrow \frac{d(v_i b(v_i))}{dv_i} == v_i$$

Hence,

$$b(v_i) = \frac{v_i}{2}$$

Dutch Auctions

- **Examples:**
  - flowers in the Netherlands, fish market in England, tobacco market in Canada
- **Typical rules:**
  - Auctioneer starts with a high price
  - Auctioneer lowers the price gradually, until some buyer shouts "Mine!"
  - The first buyer to shout "Mine!" gets the object at the price the auctioneer just called
  - Winner's profit = BV – price
  - Everyone else's profit = 0
- Dutch auctions are game-theoretically equivalent to first-price, sealed-bid auctions
  - The object goes to the highest bidder at the highest price
  - A bidder must choose a bid without knowing the bids of any other bidders
  - The optimal bidding strategies are the same

Sealed-Bid, Second-Price Auctions

- Background: Vickrey(1961)
- Used for
  - stamp collectors' auctions
  - eBay and Amazon
- **Typical rules:**
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer
  - The auctioneer opens the bid and finds the highest bidder
  - The highest bidder gets the object being sold, for a price equal to the second highest bid
  - Winner's profit = BV – price
  - Everyone else's profit = 0

Sealed-Bid, Second-Price Auctions

- Sealed-bid, 2nd-price auctions are nearly equivalent to English auctions
  - The object goes to the highest bidder
  - Price is close to the second highest BV
- and the utility functions are given by

$$U_i(b_i, b_{-i}, v_i) = \begin{cases} v_i - \max_{i \neq j} b_j & \text{if } b_i > \max_{i \neq j} b_j \\ 0 & \text{if } b_i < \max_{i \neq j} b_j \end{cases}$$

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Incomplete Information in Extensive Form Games

- Many situations of incomplete information cannot be represented as static or strategic form games.
- Instead, we need to consider extensive form games with an explicit order of moves—or dynamic games.
- In this case, we use information sets to represent what each player knows at each stage of the game.
- Since these are dynamic games, we will also need to strengthen our Bayesian Nash equilibria to include the notion of **perfection**—as in subgame perfection.
- The relevant notion of equilibrium will be **Perfect Bayesian Equilibria**, or **Perfect Bayesian Nash Equilibria**.

Dynamic Games of Incomplete Information

- A dynamic game of incomplete information consists of:
  - A set of (pure) strategies for each player i, S_i , which includes an action at each information set assigned to the player. :
  - A set of players or agents: $N = \{1, 2, ..., n\}$
  - A set of types for each player i (t_i in T_i)
  - A sequence of histories H^t at the t^{th} stage of the game, each history assigned to one of the players (or to Nature/Chance);
  - An information partition, which determines which of the histories assigned to a player are in the same information set.
  - A payoff function for each player i : $u_i(a_1,\ldots,a_n,t_1,\ldots,t_n)$
  - A (joint) probability distribution $P(t_1,\ldots,t_n)$ over types or belief about the other players

Summery

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