نظریه بازیها
Game Theory

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Saddle-point

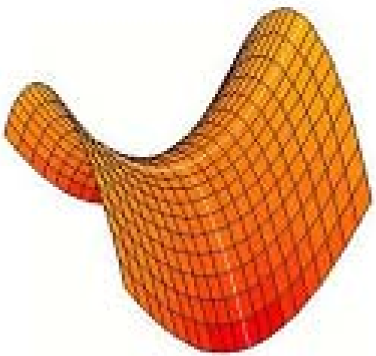
Definition: (Pure saddle-point equilibrium):

Let A define the matrix game. A pair of policies (i^*, j^*) is called a pure saddle-point equilibrium if

$$a_{i^*,j^*} \geq a_{i,j^*} \qquad \forall i \in \{1,\ldots,m\} \qquad \text{(rows - the maximizer)}$$
$$a_{i^*,j^*} \leq a_{i^*,j} \qquad \forall j \in \{1,\ldots,n\} \qquad \text{(columns - the minimizer)}$$

$$\Rightarrow a_{i,j^*} \leq a_{i^*,j^*} \leq a_{i^*,j} \qquad \forall i \in \{1,\ldots,m\}, \forall j \in \{1,\ldots,n\}$$

value a_{i^*,j^*} *is the saddle-point value;*



Saddle-point

- Row player has no incentive to deviate from his/her strategy
- Column player has no incentive to deviate from his/her strategy
- No player will regret his choice, if they both use these strategies (i^*, j^*)
- No player will benefit from an unilateral deviation from the equilibrium (Pure Nash equilibrium)

Remember :A matrix game defined by A has a saddle-point equilibrium if and only if

$$\underline{V}(A) := \max_{i \in \{1,\ldots,m\}} \min_{j \in \{1,\ldots,n\}} a_{i,j} = \overline{V}(A) := \min_{j \in \{1,\ldots,n\}} \max_{i \in \{1,\ldots,m\}} a_{i,j}$$

(i^*, j^*) is a saddle-point equilibrium

Security levels and policies

Proposition (Min-Max Property)

For every finite matrix A, the following properties hold:

(i) Security levels are well defined and unique

(ii) Both players have security policies (not necessarily unique)

(iii) The security levels always satisfy

$$\underline{V}(A) := \max_{i\in\{1,\ldots,m\}} \min_{j\in\{1,\ldots,n\}} a_{i,j} \leq \overline{V}(A) := \min_{j\in\{1,\ldots,n\}} \max_{i\in\{1,\ldots,m\}} a_{i,j}$$

Saddle-point and security levels

Proof :

Assume that $a_{i,j^*} \leq a_{i^*,j^*} \leq a_{i^*,j} \quad \forall i\ ,j$ holds

$$a_{i^*,j^*} = \min_j a_{i^*,j} \leq \max_i \min_j a_{i,j} = \underline{V}(A)$$

$$a_{i^*,j^*} = \max_i a_{i,j^*} \geq \min_j \max_i a_{i,j} = \overline{V}(A)$$

- Implies that $\overline{V}(A) \leq \underline{V}(A)$
- But from previous proposition we have $\underline{V}(A) \leq \overline{V}(A)$
- It can only be the case if $\underline{V}(A) = \overline{V}(A)$

Saddle-point and security levels

Proof: Assume that $\underline{V}(A) = a_{i^*,j^*} = \overline{V}(A)$ holds

$$a_{i^*,j^*} = \min_j \max_i a_{i,j} = \overline{V}(A)$$

$$= \max_i a_{i,j^*}$$

$$\geq a_{i,j^*} \qquad \forall i$$

$$a_{i^*,j^*} = \max_i \min_j a_{i,j} = \underline{V}(A)$$

$$= \min_j a_{i^*,j}$$

$$\leq a_{i^*,j} \qquad \forall j$$

- Implies that $a_{i,j^*} \leq a_{i^*,j^*} \leq a_{i^*,j} \qquad \forall i, j$

Election games

- Two players: Donald T. and Hillary C.
- Both players have three strategies:
  - campaign the last day in Iowa (I)
  - campaign the last day in New York (NY)
  - campaign the last day in Texas (T)

Donald T. (columns); Hilary C. (rows)

	I	NY	T
I	-3	-2	6
NY	2	0	2
T	5	-2	-4

Election games

- Hillary C. tries to maximize the voters that she wins from Donald T.
- Donald T. tries to minimize the voters that he looses to Hillary C.

Donald T.

Hilary C.

	I	NY	T
I	-3	-2	6
NY	2	0	2
T	5	-2	-4

Election games

Donald T.

Hilary C.

	I	NY	T
I	-3	-2	6
NY	2	0	2
T	5	-2	-4

-3

0

-4

- Hillary C. tries to maximize the voters
- Conservative policy for Hillary C.
  - Choose NY (Worst case gain of 0)

Election games

Donald T.

Hilary C.

	I	NY	T	
I	-3	-2	6	-2
NY	2	0	2	0
T	5	-2	-4	-4
	5	0	6	

- Donald T. tries to minimize the voters
- Conservative policy for Donald T.
  - Choose NY (Worst case loss of 0)

Election games

Donald T. (columns) / Hilary C. (rows)

	I	NY	T	
I	-3	-2	6	-2
NY	2	(0)	2	(0)
T	5	-2	-4	-4
	5	(0)	6	

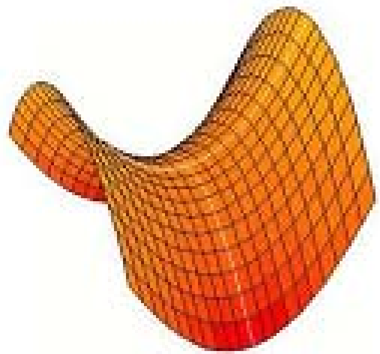
- Only rational outcome: both players play strategy NY
- They cannot expect a better outcome with any other strategy

Saddle-point equilibrium

Donald T.

Hilary C.

	I	NY	T
I	-3	-2	6
NY	2	0	2
T	5	-2	-4



The cell (NY,NY) is a saddle point:

- the cell has the smallest value in its row
- column player has no incentive to deviate from his strategy
- the cell has the larger value in its column
- row player has no incentive to deviate from her strategy

Election games (Multiple Saddle-points)

- **Consider the following setting of the Elections Game:**

Donald T.

Hilary C.

	I	NY	T	
I	-3	-2	0	-3
NY	2	0	0	0
T	5	-2	-4	-4
	5	0	0	

Order interchangeability

Suppose that a game defined by a matrix A has two distinct saddle-point equilibria: (i_1^*, j_1^*) and (i_2^*, j_2^*)

From previous Theorem, both have exactly the same value $\underline{V}(A) = \overline{V}(A)$ and

- i_1^* and i_2^* are security policies for P1
- j_1^* and j_2^* are security policies for P2

What about points (i_1^*, j_2^*) and (i_2^*, j_1^*) ?

Order interchangeability

Proposition

(Order interchangeability). If (i_1^*, j_1^*) and (i_2^*, j_2^*) are saddle-point equilibria for a matrix game defined by A, then (i_1^*, j_2^*) and (i_2^*, j_1^*) are also saddle-point equilibria for the same game and all equilibria have exactly the same value.

This property only holds for zero-sum games

Election games

- Consider the following setting of the Elections Game:

Donald T.

Hilary C.

	I	NY	T
I	1	0	-2
NY	3	-1	2
T	3	2	0

Election games

Donald T.

Hilary C.

	I	NY	T	
I	1	0	-2	-2
NY	3	-1	2	-1
T	3	2	0	0

- Conservative strategy for Hillary C.
  - Choose T (Worst case gain of 0)

Election games

Donald T.

Hilary C.

	I	NY	T	
I	1	0	-2	-2
NY	3	-1	2	-1
T	3	2	0	0
	3	2	2	

- Conservative strategy for Donald T.
  - Choose either NY or T (Worst case loss of 2)

Security v.s. Regret: Alternate play

Assume P2 plays first. Question: would P2 be happy with his choice if he solves

$$\overline{V}(A) = \min_{j\in\{1,\ldots,n\}} \max_{i\in\{1,\ldots,m\}} a_{i,j}$$

- The interpretation of the min max is that P1(the maximizer) can see the action of P2(the minimizer)
- The safe strategy for P2 is to be conservative
- He will not regret his decision
- The same argument holds for P1

These conclusions generalize to any game with alternate play : in such games, there is no reason for rational players to ever regret their decision to play a security policy.

Security v.s. Regret: Simultaneous plays

Suppose players decide actions simultaneously, i.e., without knowing the others choice.

If both players use their respective security policies then

- P1 selects row 3 (guarantees reward ≥ 0)
- P2 selects column 3 (guarantees cost ≤ 2)

This leads to cost/reward

$$0 \in [\underline{V}(A), \overline{V}(A)] = [0, 2]$$

	I	NY	T	
I	1	0	-2	-2
NY	3	-1	2	-1
T	3	2	0	0
	3	2	2	

Security v.s. Regret: Simultaneous plays

Donald T.

Hilary C.

	I	NY	T	
I	1	0	-2	-2
NY	3	-1	2	-1
T	3	2	0	0
	3	2	2	

After the game is over

- P2 is happy with choice since column 3 was the best response to row 3
- P1 regrets choice: “if I knew P2 was going to play column 3, I would have played row 2, leading to reward 2 ≥ 0”

Security v.s. Regret: Simultaneous plays

Perhaps they should have played

- P1 selects row 2
- P2 selects column 3

leading to cost/reward = 2

- Now the minimizer regrets his choice
- No further "a-posteriori" revision of their decisions would lead to a no-regret outcome

- Unlike alternate play, security policies may lead to regret in matrix games with simultaneous play

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T. (columns) / Hilary C. (rows)

	I	NY	T
I	1	2	4
NY	1	0	5
T	0	1	-1

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T. (columns); Hilary C. (rows)

	I	NY	T
I	1	2	4
NY	1	0	5
T	0	1	-1

Observation:

- Strategy1(“Iowa”) is always better for Hillary C. than strategy 3 (“Texas”)

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T. (columns); Hilary C. (rows)

	I	NY	T
I	1	2	4
NY	1	0	5
T	0	1	-1

Observation:

- Strategy1 (“Iowa”) is always better for Hillary C. than strategy 3 (“Texas”)

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T. (columns); Hilary C. (rows)

	I	NY	T
I	1	2	4
NY	1	0	5

Observation:

- Hillary C. will never play strategy3; we can remove it

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T. (columns), Hilary C. (rows)

	I	NY	T
I	1	2	4
NY	1	0	5

Observation:

- Both strategy1 (“Iowa”) and strategy2 (“New York”) are always better for Donald T. than strategy3 (“Texas”)

Dominant Strategy Equilibria

- Consider the following setting of the Elections Game:

Donald T. (columns), Hilary C. (rows)

	I	NY
I	1	2
NY	1	0

Observation:

- Donald T. will never play strategy3; we can remove it

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T.

Hilary C.

	I	NY
I	1	2
NY	1	0

Observation:

- Strategy 1 (“Iowa”) is always better for Hillary C. than strategy 2 (“NewYork”).

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T.

Hilary C.

	I	NY
I	1	2

Observation:

- Hillary C. will never play strategy 2; we can remove it

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T.

Hilary C.

	I	NY
I	1	2

Observation:

- Strategy 1 (“Iowa”) is always better for Donald T. than strategy 2 (“NewYork”).

Dominant Strategy Equilibria

- **Consider the following setting of the Elections Game:**

Donald T.

Hilary C.

	l
l	1

Observation:

- Donald T. will never play strategy 2; we can remove it
- Conclusion: Both Donald T. and Hillary C. campaign in Iowa.

Dominant Strategy Equilibria

- Consider the following setting of the Elections Game:

Donald T. (columns); Hilary C. (rows)

	I	NY	T	
I	1	2	4	1
NY	1	0	5	0
T	0	1	-1	-1
	1	2	5	

Strictly Dominating Policies

- We say that row i strictly dominates row k if

$$a_{i,j} > a_{k,j} \qquad \forall j = \{1,\ldots,\mathrm{n}\}$$

- which means that no matter what P2does, the maximizer P1 is always better off by selecting row i instead of row k

P1 will never select row k

Same holds true for column dominance

Weakly Dominating Policies

- We say that row i weakly dominates row k if

$$a_{i,j} \geq a_{k,j} \qquad \forall j = \{1, \ldots, \mathrm{n}\}$$

- which means that no matter what P2does, the maximizer P1 loses nothing by selecting row i instead of row k

P1 can ignore row k without losing anything

Same holds true for column weak dominance

Pareto Optimality

Definition: Pareto Optimality

An outcome is Pareto optimal if there is no other outcome which would give both players a higher payoff or would give one player the same payoff and the other player a higher payoff

	A	B	C
X	1,3	0,0	-1,890
Y	2,2	0,2	0,4
Z	-1,4	-5,5	-1,1

Pareto Optimality

Pareto Optimality

	A	B	C
X	1,3	0,0	-1,890
Y	2,2	0,2	0,4
Z	-1,4	-5,5	-1,1

Pareto Optimality

	A	B	C
X	1,3	0,0	-1,890
Y	2,2	0,2	0,4
Z	-1,4	-5,5	-1,1

Pareto Optimality

Pareto Optimality

	A	B	C
X	1,3	0,0	-1,890
Y	2,2	0,2	0,4
Z	-1,4	-5,5	-1,1