نظریه بازیها
Game Theory

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Zero-Sum Matrix Games (Mixed Strategies)

Material

- Dynamic Non-cooperative Game Theory: Second Edition
  - Chapter 2. 2 and 2. 3
- An Introductory Course in Non-cooperative Game Theory
  - Chapter 4, 6

Recap

Setting of Two-Person Zero-Sum Games:

two players (“**row player**” and “**column player**”)

- row player chooses one out of **m** strategies, column player chooses one out of n strategies

Payoff matrix

- If row player plays i and column player plays j
- then, row player gains a_{ij} and column player looses a_{ij}
- row player maximizes
- column player minimizes

Column player’s actions

Row player’s actions

a_{11}	a_{12}	…	a_{1n}
a_{21}	a_{22}	…	a_{2n}
…			
a_{m1}	a_{m2}	…	a_{mn}

Zero-Sum Property: One player wins whatever other player loses

Saddle-point

Definition: (Pure saddle-point equilibrium):

Let A define the matrix game. A pair of policies (i^*, j^*) is called a pure saddle-point equilibrium if

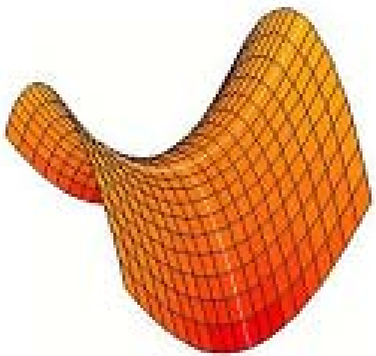
$$a_{i^*,j^*} \geq a_{i,j^*} \qquad \forall i \in \{1,\ldots,m\} \qquad \text{(rows - the maximizer)}$$
$$a_{i^*,j^*} \leq a_{i^*,j} \qquad \forall j \in \{1,\ldots,n\} \qquad \text{(columns - the minimizer)}$$

$$\Rightarrow a_{i,j^*} \leq a_{i^*,j^*} \leq a_{i^*,j} \qquad \forall i \in \{1,\ldots,m\}, \forall j \in \{1,\ldots,n\}$$

value a_{i^*,j^*} *is the saddle-point value;*

- Saddle-point and security levels

  $\underline{V}(A) = a_{i^*,j^*} = \overline{V}(A)$



Election games

- Consider the following setting of the Elections Game:
- Both players have three strategies:
  - campaign the last day in Iowa (I)
  - campaign the last day in New York (NY)
  - campaign the last day in Texas (T)

Donald T. (columns) / Hilary C. (rows)

	I	NY	T
I	1	0	-2
NY	3	-1	2
T	3	2	0


Election games

Donald T.

Hilary C.

	I	NY	T	
I	1	0	-2	-2
NY	3	-1	2	-1
T	3	2	0	0

- Conservative strategy for Hillary C.
  - Choose T (Worst case gain of 0)

Election games

Donald T.

Hilary C.

	I	NY	T	
I	1	0	-2	-2
NY	3	-1	2	-1
T	3	2	0	0
	3	2	2	

- Conservative strategy for Donald T.
  - Choose either NY or T (Worst case loss of 2)
  - No Pure Nash Equilibrium exist

"Odds-and-Evens" Game

Example: "Odds-and-Evens" Game

- if sum of both numbers is even: P_1 wins 1 Euro, P_2 looses 1 Euro
- if sum of both numbers is odd: P_2 wins 1 Euro, P_1 looses 1 Euro
- P_1 maximizes
- P_2 minimizes

P_2 (columns), P_1 (rows)

	odd	even	
odd	1	-1	-1
even	-1	1	-1
	1	1	

- No Pure Nash Equilibrium exist
- $\underline{V}(A) = -1 < 1 = \overline{V}(A) \Rightarrow$ No pure saddle point equilibrium

“Odds-and-Evens” Game

P_2

P_1		odd	even
	odd	1	-1
	even	-1	1

Suppose you are playing “Odds-and-Evens” with a mind reader

- How do you avoid losing?
- Answer: Don’t think! Instead flip a coin
  - At best, the mind reader can win half the time

"Odds-and-Evens" Game

- Intuitively, flipping a coin, i.e., playing 50% and 50% , introduces a third option for both players

P_2

P_1

	odd	even	50%/ 50%
odd	1	-1	0
even	-1	1	0
50%/ 50%	0	0	0

"Odds-and-Evens" Game

- $\underline{V}(A') = 0 = \overline{V}(A') \Rightarrow$ a saddle point equilibrium exists

  $(50\% / 50\%, 50\% / 50\%)$

P_2

P_1

	odd	even	50%/ 50%	
odd	1	-1	0	-1
even	-1	1	0	-1
50%/ 50%	0	0	0	0
	1	1	0	

Example:

Consider the following game:

- P_1 maximizes
- P_2 minimizes

P_2

P_1

	up	down	
left	3	-1	-1
right	-2	0	-2
	3	0	

- $\underline{V}(A) = -1 < 0 = \overline{V}(A) \Rightarrow$ No pure saddle point equilibrium
- Assume that both players flip the coin. Does it work in this setting?

"Odds-and-Evens" Game

P_2

P_1		up	down	50%/ 50%	
	left	3	-1	1	-1
	right	-2	0	-1	-2
	50%/ 50%	0.5	-0.5	0	-0.5
		3	0	1	

- $\underline{V}(A') = -0.5 < 0 = \overline{V}(A')$.
- Notice that $(\frac{1}{2}\,left\,/\,\frac{1}{2}\,\text{right}, \frac{1}{2}\,up\,/\,\frac{1}{2}\,down)$ is not a saddle point:
- P_1 can do better by switching to right
- P_2 can do better by switching to down
- What probabilities should each player use?

Example:

- Find a strategy for P_1 that makes P_2 in different from selecting up or down
- Find a strategy for P_2 that makes P_1 indifferent from selecting left or right

P_2

P_1		Up z	Down 1-z
	Left y	3	-1
	Right 1-y	-2	0

Example:

- Find a strategy for P_1 , i.e., find y that makes P_2 indifferent from selecting up or down

P_2 (columns), P_1 (rows)

	Up z	Down 1-z
Left y	3	-1
Right 1-y	-2	0

- $E(P_u) = E(P_d)$ (expected payoff of game given that P_2 plays up = expected payoff of game given that P_2 plays down)
- $E(P_u) = g(y)$ (expected payoff for left is a function of y)
- $E(P_d) = g(y)$ (expected payoff for right is a function of y)

Example:

- Find a strategy for P_2 , i.e., find z that makes P_1 indifferent from selecting left or right

P_1 \ P_2	Up z	Down 1-z
Left y	3	-1
Right 1-y	-2	0

$$E(P_u) = g(y) = 3y + (-2)(1-y) = 5y - 2$$
$$E(P_d) = g(y) = -1y + 0(1-y) = -y$$
$$E(P_u) = E(P_d) = 5y - 2 = -y$$
$$y = 1/3$$

Example:

- Find a strategy for P_2 , i.e., find z that makes P_1 indifferent from selecting left or right

P_1 \ P_2	Up z	Down 1-z
Left y	3	-1
Right 1-y	-2	0

- $E(P_l) = E(P_r)$(expected payoff of game given that P_1 plays left = expected payoff of game given that P_1 plays right)
- $E(P_l) = f(z)$ (expected payoff for left is a function of z)
- $E(P_r) = f(z)$ (expected payoff for right is a function of z)

Example:

- Find a strategy for P_2 , i.e., find z that makes P_1 indifferent from selecting left or right

P_2 (columns), P_1 (rows)

	Up z	Down 1-z
Left y	3	-1
Right 1-y	-2	0

$$E(P_l) = f(z) = 3z + (-1)(1-z) = 4z - 1$$

$$E(P_r) = f(z) = -2z + 0(1-z) = -2z$$

$$E(P_l) = E(P_r) = 4z - 1 = -2z$$

$$z = \frac{1}{6}$$

Example:

- What is the expected payoff for the two players?

	P_2 Up z	Down 1-z
P_1 Left y	3	-1
Right 1-y	-2	0

- Expected payoff for both players

$$= 3\,\tfrac{1}{3}\,\tfrac{1}{6} + (-1)\,\tfrac{1}{3}\,\tfrac{5}{6} + (-2)\,\tfrac{2}{3}\,\tfrac{1}{6} + (0)\,\tfrac{2}{3}\,\tfrac{5}{6}$$

$$= -\tfrac{1}{3}$$

Example:

- Now the game has a saddle point equilibrium

  $(\frac{1}{3} \textit{left} / \frac{2}{3} \text{right}, \frac{1}{6} \textit{up} / \frac{5}{6} \textit{down})$

- $\underline{V}(A') = -\frac{1}{3} = \overline{V}(A')$
- No player can do better by unilaterally changing their strategy

(Nash equilibrium)

P_2 (columns), P_1 (rows)

	Up z	Down 1-z
Left y	3	-1
Right 1-y	-2	0

Mixed Strategies

In mixed strategies:

• the players select their actions randomly according to a previously selected probability distribution

Column player's actions

Row player's actions

	z_1	z_2	…	z_n
y_1	a_{11}	a_{12}	…	a_{1n}
y_2	a_{21}	a_{22}	…	a_{2n}
	…			
y_m	a_{m1}	a_{m2}	…	a_{mn}

Mixed Strategies

A mixed policy for P_1 is a set of numbers

$$Y = \{(y_1,\ldots,y_m) : \sum_{i=1}^{m} y_i = 1,\ y_i \ge 0,\ i = 1,\ldots,m\ \}$$

Where y_i is the probability that P_1 uses to select the action $i \in \{1,\ldots,m\}$

Similarly, A mixed policy for P_2 is a set of numbers

$$Z = \{(z_1,\ldots,z_n) : \sum_{j=1}^{n} z_j = 1,\ z_j \ge 0,\ j = 1,\ldots,n\ \}$$

Where z_j is the probability that P_2 uses to select the action $j \in \{1,\ldots,\mathrm{n}\}$

- The sets Y and Z are called the probability simplexes
- Pure policies still exists within the mixed action space

Mixed Strategies

- **Objective (mixed policies):** The player P_1 wants to maximize the expected outcome $J = y^T A z$ and the player P_2 wants to minimize the same quantity.

- There are two common interpretations for mixed policies:

- In the repeated game paradigm, the same two players face each other multiple times. In each game they choose their actions randomly according to pre-selected mixed policies (in-dependently from each other and independently from game to game) and their goal is to minimize/maximize the cost/reward averaged over all the games played. This paradigm makes sense in many games in economics, e.g., in advertising campaigns or the tax-payers auditing; and also in political/social "engineering"

Mixed Strategies

- In the large population paradigm, there is large population of players P_1 and another equally large population of players P_2. All players only play pure policies, but the percentage of players that play each pure policy matches the probabilities of the mixed policies. Two players are then selected randomly from each population (independently) and they play against each other. The goal is to select a "good mix" for the populations so as to minimize/maximize the expected cost/reward. This paradigm also makes sense in some of the above examples, e.g., tax auditing, or workers compensation. In addition, it makes sense to some robust design problems.

Mixed Strategies

The average security level for P_1 (the maximizer) is defined by

$$\underline{V}_m(A) := \max_{y \in Y} \min_{z \in Z} y^T A z$$

$$= \max_{y \in Y} \min_{z \in Z} \sum_{i=1}^{m} \sum_{j=1}^{n} y_i z_j a_{ij}$$

$$= \max_{y \in Y} \min_{z \in Z} \mathrm{E}_{\mathrm{P}}(a_{ij})$$

Where $\mathrm{P}((P_1 \ playing \ i) \cap (P_2 \ playing \ j)) = y_i z_j \quad for \ \forall i, j$

The mixed security policy is

$$y^* \in \arg\max_{y \in Y} \min_{z \in Z} y^T A z$$

Mixed Strategies

The average security level for P_2 (the minimizer) is defined by

$$\bar{V}_m(A) := \min_{z\in Z} \max_{y\in Y} y^T A\, z$$

$$= \min_{z\in Z} \max_{y\in Y} \sum_{i=1}^{m} \sum_{j=1}^{n} y_i z_j a_{ij}$$

$$= \min_{z\in Z} \max_{y\in Y} \mathrm{E}_{\mathrm{P}}(a_{ij})$$

Where $\mathrm{P}((P_1 \ playing\ i) \cap (P_2 \ playing\ j)) = y_i z_j \quad for\ \forall i, j$

The mixed security policy is

$$z^* \in \arg\min_{z\in Z} \max_{y\in Y} y^T A\, z$$

Mixed Strategies

Definition: (Mixed saddle-point equilibrium):

A pair of policies defined through the probabilities $(y^*, z^*) \in Y \times Z$ is called a mixed saddle-point equilibrium if

$$y^{*T} A z^* \geq y^T A z^* \qquad \forall y \in Y \qquad \text{(the maximizer)}$$

$$y^{*T} A z^* \leq y^{*T} A z \qquad \forall z \in Z \qquad \text{(the minimizer)}$$

and $y^{*T} A z^*$ is the is called the saddle point value.

This is also called a Nash Equilibrium: no player can do better by **unilaterally** changing his strategy (includes both pure and mixed strategies)

Mixed Strategies

Proposition (Min-Max Property)

For every finite matrix A, the following properties hold:

(i) Average security levels are well defined and unique

(ii)Both players have mixed security policies (not necessarily unique)

(iii) The security levels always satisfy

$$\underline{V}(A) \leq \underline{V}_m(A) \leq \overline{V}_m(A) \leq \overline{V}(A)$$

Min-Max Property

Proof :

$$\underline{V}_m(A) := \max_{y \in Y} \min_{z \in Z} y^T A z$$

$$\geq \max_{y \in \{e_1, \ldots, e_m\}} \min_{z \in Z} y^T A z$$

Where $e_i = (0, \ldots, 1, \ldots, 0)$ is the i^{th} canonical basis in R^n

follows from restricting the feasible region of y, since $\{e_1, \ldots, e_m\} \subset Y$

Min-Max Property

Proof :

$$\underline{V}_m(A) := \max_{y \in Y} \min_{z \in Z} y^T A z$$

$$\geq \max_{y \in \{e_1, \ldots, e_m\}} \min_{z \in Z} y^T A z$$

$$= \max_{i \in \{1, \ldots, m\}} \min_{z \in Z} e_i^T A z$$

$$= \max_{i \in \{1, \ldots, m\}} \min_{z \in Z} \begin{bmatrix} a_{i1} & \ldots & a_{in} \end{bmatrix} z$$

$$= \max_{i \in \{1, \ldots, m\}} \min_{z \in Z} \sum_{j=1}^{n} z_j a_{ij}$$

Min-Max Property

From Optimal control ,we know that if X is a simplex, then

$$\min_{x} \sum_{j=1}^{n} x_j \beta_j \quad \Leftrightarrow \min_{j\in\{1,\dots,n\}} \beta_{ij}$$

$$s.t. \;\; x \in X$$

Using that

$$\underline{V}_m(A) := \max_{y\in Y} \; \min_{z\in Z} \; y^T A z$$

$$\geq \max_{y\in\{e_1,\dots,e_m\}} \; \min_{z\in Z} \; y^T A z$$

$$= \max_{i\in\{1,\dots,m\}} \; \min_{z\in Z} \sum_{j=1}^{n} z_j a_{ij}$$

$$= \max_{i\in\{1,\dots,m\}} \; \min_{j\in\{1,\dots,n\}} \; a_{ij} = \underline{V}(A)$$

Proof for $\bar{V}_m(A) \leq \bar{V}(A)$ follows using similar arguments

Min-Max Property

Proof for From $\underline{V}_m(A) \le \overline{V}_m(A)$, let y^* be a mixed security policy for P$_1$ (the maximizer)

$$\underline{V}_m(A) := \max_{y \in Y} \min_{z \in Z} y^T A\, z$$

$$= \min_{z \in Z} y^{*T} A\, z$$

Notice that for any vector v, since $y^* \in Y$,then

$$y^{*T} v \le \max_{y \in Y} y^T v$$

Letting $v = Az$, then

$$\underline{V}_m(A) := \max_{y \in Y} \min_{z \in Z} y^T A\, z$$

$$= \min_{z \in Z} y^{*T} A\, z$$

$$\le \min_{z \in Z} \max_{y \in Y} y^T A\, z := \overline{V}_m(A)$$

Mixed Strategies

Theorem (Mixed saddle-point vs. security levels)

A matrix game defined by A has a **mixed saddle-point equilibrium** if and only if

$$\underline{V}_m(A) = \max_{y \in Y} \min_{z \in Z} y^T A z = \min_{z \in Z} \max_{y \in Y} y^T A z = \overline{V}_m(A)$$

In particular,

- (y^*, z^*) is a mixed saddle-point equilibrium
- $\underline{V}_m(A) = \overline{V}_m(A)$ is the saddle point value

This condition holds for all matrices A

→ For any two **player zero-sum game** there exists a saddle point equilibrium (Nash equilibrium)