# نظریه بازیها
# Game Theory

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# Non-zero sum games

Material

- Dynamic Non-cooperative Game Theory: Second Edition
  - Chapter 3.1,3.2
- An Introductory Course in Non-cooperative Game Theory
  - Chapter 9

# Zero sum games

- Zero sum games
  - Definitions
  - Pure strategies
  - Dominating strategies
  - Mixed strategies
  - Linear programs to find NE
  - Extensive form and information structure
  - Feedback games: pure, mixed, and behavioral strategies
- Non-zero sum games
  - N-player games
  - Bimatrix formulation

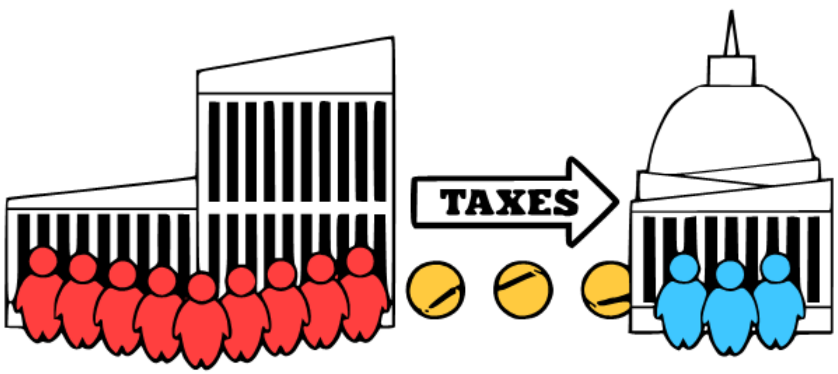
# Non-zero-sum games

In **non-zero-sum games**, player could have

- different, but not opposite, interests (cost functions)
- even the same interest!

Example: Economics

- The **government** chooses income tax rates to maximize some (arbitrary) fairness criterion.
- Individual **taxpayers** try to maximize their wealth (not necessarily reducing taxes).
  - Work 100% and pay 25% taxes, or
  - Work 60% and pay 15% taxes?



# Non-zero-sum games

Example: Business

- Two car companies must decide whether to buy ads.
  Iran car marketing: $27 million/day.
- Displaying ads gives you a marketing advantage.
- Total sales are marginally affected by advertising.


Company 2

| Company 1 | No Ads | Ads |
|---|---|---|
| No Ads | 2,2 | 1,2.5 |
| Ads | 2.5,1 | 1.5,1.5 |

# Non-zero-sum games

**Two or more** players

For each Player i :

- Strategy $\gamma^{(i)} \in \Gamma^{(i)}$
- Outcome $J_i(\gamma^{(1)},\ldots,\gamma^{(N)})$
- All players are minimizers

Other assumptions continue to hold:

- All agents are rational and assume others are rational
- All agents know the cost function of other players
  - Otherwise, an interesting learning problem: exploration vs exploitation

# Security levels

**Least cost** assuming the worst possible choice of the other players

$$\bar{V}_i := \min_{\gamma^{(i)}\in\Gamma^{(i)}} \max_{\gamma^{(-i)}\in\Gamma^{(-i)}} J_i(\gamma^{(i)},\gamma^{(-i)})$$

Notation: ***-i*** refers to all Players but $i$. E.g. $\gamma^{(-i)} = \{\gamma^{(j)}\}_{j\neq i}$

- Security level for Player $i$ depends only on $J_i$
- Worst case assumes non-rational play by other Players
  - Other Players minimize their costs, they don't maximize $J_i$
- Security level $\bar{V}$ does not describe the outcome of a sequential game where Player *i* plays first, and the others respond
- A very conservative bound!

# Nash equilibrium

$\gamma^{(1)*},\ldots,\gamma^{(N)*}$ is a Nash equilibrium if, for any Player *i*,

$$J_i(\gamma^{(i)},\gamma^{(-i)*}) \geq J_i(\gamma^{(i)*},\gamma^{(-i)*})$$

for all $\gamma^{(i)} \in \Gamma^{(i)}$

- No "saddle-point" characterization
- No-regret interpretation continues to hold
  - No Player has an incentive to deviate from the NE strategy, if all the others are playing their NE strategies
- The definition of Nash equilibrium depends on all $J_i$'s

# Two players:

Special case: two players.

- Player1 $\Gamma^{(1)} = \{\gamma_1^{(1)}, \dots, \gamma_m^{(1)}\}$
- Player2 $\Gamma^{(2)} = \{\gamma_1^{(2)}, \dots, \gamma_n^{(2)}\}$

**Bimatrix representation**

$$A_{ij} := J_1(\gamma_i^{(1)}, \gamma_j^{(2)}) \qquad B_{ij} := J_2(\gamma_i^{(1)}, \gamma_j^{(2)})$$

$$A := \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \dots & \vdots \\ A_{m1} & \dots & A_{mn} \end{bmatrix} \qquad B := \begin{bmatrix} B_{11} & \dots & B_{1n} \\ \vdots & \dots & \vdots \\ B_{m1} & \dots & B_{mn} \end{bmatrix}$$

- $A = -B$ zero sum game
- $A = B$ is allowed: Players with identical interest

## Example:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \qquad B := \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

- **Security level $P_1$**

$$\overline{V}_1 = \min_i \max_j A_{ij} = \min\{1,2\} = 1$$

(obtained via the security policy $i = 1$)

- **Security level Player 2:**

$$\overline{V}_2 = \min_j \max_i B_{ij} = \min\{2,3\} = 2$$

(obtained via the security policy $j = 1$)

Each Player can compute $\overline{V}_i$ without knowledge of the other.

# Example:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \qquad B := \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

- **Nash equilibria**
  - (1,1) is a NE with outcome (1,2)
  - (2,2) is a NE with outcome (−1,0)
- In this case, the security policies $(i,j) = (1,1)$ correspond to a Nash equilibrium
- However, we have also another Nash equilibrium which is not a security policy
- Different Nash equilibria have different outcomes
- Nash equilibria cannot be interchanged,

e.g.(1,2)and(2,1)are not Nash equilibria

# Example:(Battle of sex)

Consider the bimatrix game defined by the following matrices:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \qquad B := \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

- **Security level $P_1$**

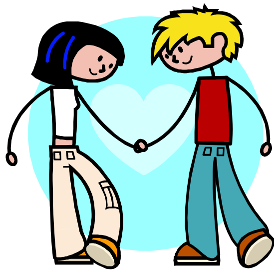
$$\bar{V}_1 = \min_i \max_j A_{ij} = \min\{1,0\} = 0$$

(obtained via the security policy $i = 2$)

- **Security level Player 2:**

$$\bar{V}_2 = \min_j \max_i B_{ij} = \min\{2,3\} = 2$$

(obtained via the security policy $j = 1$)

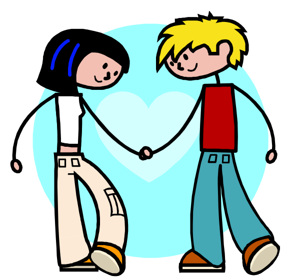


# Example:(Battle of sex)

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \qquad B := \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

- **Nash equilibria**
  - (1,1) is a NE with outcome (-2,-1)
  - (2,2) is a NE with outcome (−1,-2)
- In this case, the security policies $(i,j) = (2,1)$ are not always Nash equilibrium
- Different Nash equilibria have different outcomes
- Nash equilibria cannot be interchanged,

e.g.(1,2)and(2,1)are not Nash equilibria



# Multiple Nash equilibria

A fundamental question that didn't arise in zero-sum games.

**Multiple NE**

Assume a game has multiple Nash equilibria, with different outcomes.

What NE strategy will each player play?.

- Are some Nash equilibria "preferrable"?
- Example 1: (1,2) and(-1,0)
- Example 2: (-2,-1)and(-1,-2)

**Remember:** Nash equilibria are not interchangeable!

$(-1,0) \prec (1,2)$ but $(-2,-1) \neq (-1,-2)$

$(-2,-1) \neq (-1,-2)$

# Admissible Nash equilibria

A Nash equilibrium $(\gamma^{(1)}, \gamma^{(2)})$ is admissible if there is no other Nash equilibrium $(\tilde{\gamma}^{(1)}, \tilde{\gamma}^{(2)})$ such that

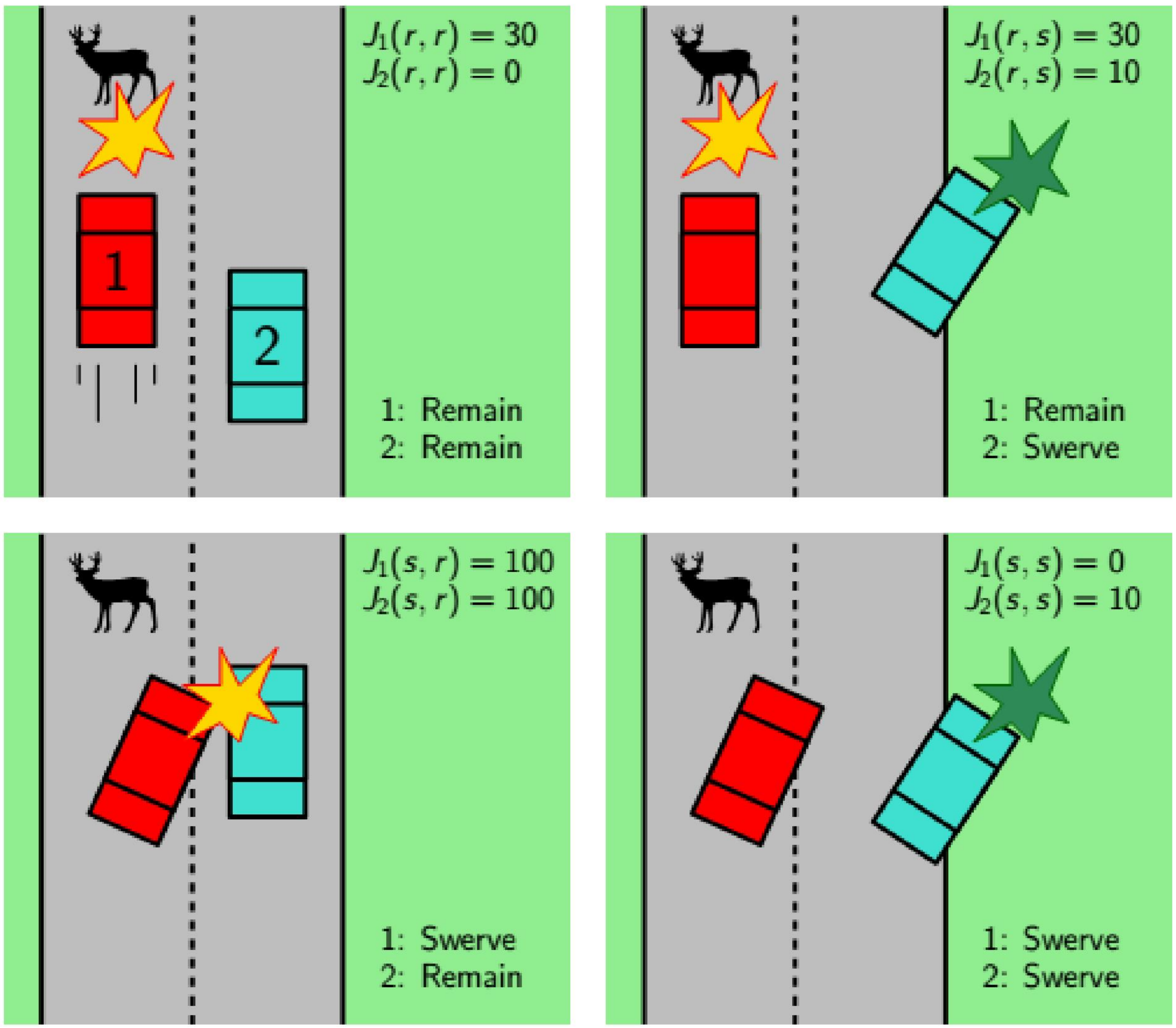
$$J_1(\tilde{\gamma}^{(1)}, \tilde{\gamma}^{(2)}) \le J_1(\gamma^{(1)}, \gamma^{(2)}) \quad \text{and} \quad J_2(\tilde{\gamma}^{(1)}, \tilde{\gamma}^{(2)}) \le J_2(\gamma^{(1)}, \gamma^{(2)})$$

**with at least one of the two inequalities strict.**

- Example 1: (1,2) and(-1,0) $\qquad (-1,0) \prec (1,2)$
- Example 2: (-2,-1)and(-1,-2) $\qquad (-2,-1) \neq (-1,-2)$

$$(-2,-1) \neq (-1,-2)$$

# Example:self driving car



# Example: self driving car

$$A = \begin{bmatrix} 30 & 30 \\ 100 & 0 \end{bmatrix} \qquad B := \begin{bmatrix} 0 & 10 \\ 100 & 10 \end{bmatrix}$$

- Security strategy Player 1: remain, $\bar{V}_1 = \min\{30, 100\} = 30$
- Security strategy Player 2: swerve, $\bar{V}_2 = \min\{100, 10\} = 10$
- Nash equilibria (no regret strategy)
  - (remain,remain), outcome (30, 0)
  - (swerve,swerve), outcome (0, 10)
- Admissible Nash equilibria?

**Both**

# Example: self driving car

- Which strategy will the two cars play?

Very hard to predict!

- Multiple admissible Nash equilibria which
  - have **different values**
  - are not **interchangeable**

$$(30,0) \neq (0,10)$$


$$(30,0) \neq (0,10)$$

# Example: self driving car

Few options available:

1)Both Players play their security strategy

- (remain, swerve), with outcome (30, 10) (worse than both NE…)

2) Mechanism design, i.e. change the Players' costs to induce a unique admissible Nash equilibrium

- example: a fine of 30 is you cross lanes and hit another car

$$(30,0) \prec (30,10)$$

# Interchangeable Nash equilibria

The lack of interchangeability is an "unpleasant" possibility in nonzero-sum games and leads to the following hierarchy of nonzero-sum games:

1)Games with **a unique NE**, or multiple interchangeable NE

**Easy to predict the behavior of players**, or at least the resulting outcome.

2) Games with a **unique admissible NE**, or multiple interchangeable admissible NE

**Still quite predictable**, once the non admissible NE are eliminated.

3 ) games with multiple admissible Nash equilibrium that are interchangeable but have **different values**,

- noncooperative **rational players** will likely end up in a Nash equilibria, but it will generally be **difficult to predict which**.

# Interchangeable Nash equilibria

4 )Game with multiple admissible NE that are not interchangeable

**Difficult to predict whether** players will agree on a NE strategy to play.

When played repeatedly, these games can lead to persistent oscillations in the policies used by the players as they try to adjust to the most recent policy used by the other player.

- The players may simply use security policies, leading to minimax solutions. Such solutions are often costly for both players and therefore not efficient.
- When possible, the reward structure of the game should be changed to avoid inefficient solutions and policy oscillations in repeated games.

It is often possible to "reshape" the reward structure of a game in economics (and engineering) through pricing, taxation, or other incentives/deterrents.

# Recap :Mixed Strategies

In mixed strategies:

- the players select their actions randomly according to a previously selected probability distribution

A mixed policy for $P_1$ is a set of numbers

$$Y = \{(y_1, \ldots, y_m) : \sum_{i=1}^{m} y_i = 1,\ y_i \geq 0,\ i = 1, \ldots, m\ \}$$

- A mixed policy for $P_2$ is a set of numbers

$$Z = \{(z_1, \ldots, z_n) : \sum_{j=1}^{n} z_j = 1,\ z_j \geq 0,\ j = 1, \ldots, n\ \}$$

# Mixed Strategies

- It is assumed that the random selections by **both players** are done **statistically independently** and that the players try to **minimize** the resulting **expected outcome** J of the game
- for a particular pair of mixed policies $(y,z) \in Y \times Z$ for $P_1$ and $P_2$, respectively,
  - $J_1 = y^T A z$ is the outcome for $P_1$
  - $J_2 = y^T B z$ is the outcome for $P_2$

# Mixed Nash Equilibrium

**Definition: (Mixed Nash Equilibrium equilibrium):**

A pair of policies defined through the probabilities $(y^*, z^*) \in Y \times Z$ is called a mixed Nash equilibrium if

$$y^{*T} A z^* \leq y^T A z^* \qquad \forall y \in Y$$

$$y^{*T} B z^* \leq y^{*T} B z \qquad \forall z \in Z$$

and $(y^{*T} A z^*, y^{*T} B z^*)$ is the is called the Nash outcome of the game.

- As in zero-sum games, the introduction of mixed policies enlarged the action spaces for both players to the point that Nash equilibria now always exist:
- **Theorem (Nash).** Every bimatrix game has at least one mixed Nash equilibrium.

# Mixed Nash equilibrium (full generality)

The strategies $(y^{(1)*},\ldots,y^{(N)*})$ describe a mixed Nash equilibrium for a non-zero-sum N person game if, for any Player $i$,

$$J_i(y^{(i)}, y^{(-i)*}) \geq J_i(y^{(i)*}, y^{(-i)*})$$

for all $y^{(i)} \in Y^{(i)}$

**Theorem (Nash) (full generality)**

- Every N -person, non-zero-sum game, has at least one mixed Nash equilibrium