# نظریه بازیها
# Game Theory

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# Non-zero sum games

Material

- Dynamic Non-cooperative Game Theory: Second Edition
  - Chapter 3.4
- An Introductory Course in Non-cooperative Game Theory
  - Chapter 10

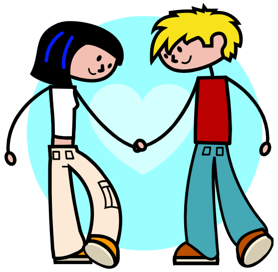
# Non-zero sum games

- ☐ Zero sum games
- ☐ Non-zero sum games
  - ☐ N-player games
  - ☐ Bimatrix formulation
  - ☐ Nash equilibrium in mixed strategies
  - ☐ Completely mixed NE
  - ☐ Computing mixed NE

# Example:(Battle of sexes)

Consider a different version of the battle of sexes is defined by the following matrices:

$$A=\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \qquad B:=\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$



- ❑ To find a mixed NE , we need to compute vectors

$$y^{*} := [y^{*}_{1} \quad 1-y^{*}_{1}]^{T} \qquad , y^{*}_{1} \in [0,1]$$

$$z^{*} := [z^{*}_{1} \quad 1-z^{*}_{1}]^{T} \qquad , z^{*}_{1} \in [0,1]$$

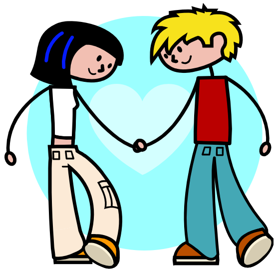
For which

$$y^{*\mathrm{T}} \mathrm{A} z^{*} = y^{*}_{1}(1-6z_{1}^{*})+4z_{1}^{*}-1 \le y_{1}(1-6z_{1}^{*})+4z_{1}^{*}-1, \qquad y_{1} \in [0,1] (1)$$

$$y^{*\mathrm{T}} B z^{*} = z^{*}_{1}(2-6y_{1}^{*})+5y_{1}^{*}-2 \le z_{1}(2-6y_{1}^{*})+5y_{1}^{*}-2, \qquad z_{1} \in [0,1] (2)$$

# Example:(Battle of sexes)

This can be achieved if we are able to make the right hand side of (1) independent of $y_1$ and the right hand side of (2) independent of $z_1$ . In particular by making



$$(1-6z_1^*)=0 \qquad \Leftrightarrow z_1^* = {}^1\!/_6$$
$$(2-6y_1^*)=0 \qquad \Leftrightarrow y_1^* = {}^1\!/_3$$

- This leads to following mixed Nash equilibrium and outcomes

$$(y^*,z^*)=([{}^1\!/_3 \quad {}^2\!/_3]^T, \quad [{}^1\!/_6 \quad {}^5\!/_6]^T)$$
$$(y^{*T}Az^*, y^{*T}Bz^*)=(4z^*{}_1-1,5y_1^*-2) \;\; z=(-{}^1\!/_3,-{}^1\!/_3)$$

And therefore it is also a NE for a bimatrix game defined by the matrices (-A,-B), which correspond to exactly opposite objectives by both players

# Completely Mixed Nash Equilibrium

**Definition:** A Nash equilibrium $(y^*, z^*) \in Y \times Z$ is completely mixed or an inner-point equilibria if all probabilities are strictly positive, i.e.

$$y_i > 0, \quad \forall i = 1, \ldots, m \quad and \quad z_j > 0, \quad \forall j = 1, \ldots, n$$

**Lemma:** If $(y^*, z^*)$ is a completely mixed Nash equilibria with outcomes $(p^*, q^*)$ for a bimatrix game defined by the matrices (A, B), then $Az^* = p^* 1_{m\times 1}$ $\qquad B^T y^* = q^* 1_{n\times 1}$

Consequently, $(y^*, z^*)$ is also a mixed Nash equilibria for the three bimatrix games defined by (-A, -B), (A, -B), and (-A, B).

# Proof Completely Mixed NE

Assuming that $(y^*, z^*)$ is a completely mixed Nash equilibrium for the game defined by (A, B), we have that

$$y^{*T} A\, z^* = \min_y \; y^T A\, z^* = \min_y \sum_i y_i \,(A\, z^*)_i$$

**ith row of Az***

- ☐ If one row $i$ of $Az^*$ was strictly larger than any of the remaining ones, then the minimum would be achieved with $y_i = 0$ and the Nash equilibria would not be completely mixed. Therefore to have a completely mixed equilibria, we need to have all the rows of $Az^*$ exactly equal to each other:

$$Az^* = p^* 1_{m\times 1}$$

# Proof Completely Mixed NE

for some scalar $p*$, which means that

$$y^{*T} A z^* = y^T A z^* = p^*, \quad \forall y, y^* \in Y$$

- ☐ Similarly, since none of the $z_i = 0$, we can also conclude that all columns of $y^{*T} B$ (which are the rows of $B^T y^*$) must all be equal to some constant $q*$ and therefore

$$y^{*T} B z^* = y^{*T} B z = q^*, \quad \forall z, z^* \in Z$$

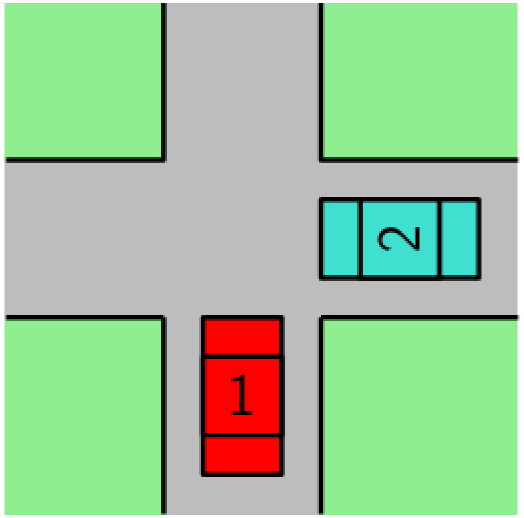
- ☐ we conclude $(p^*, q^*)$ is indeed the Nash outcome of the game and that $(y^*, z^*)$ *is* also a mixed Nash equilibrium for the three bimatrix games defined by *(-A, -B)*, *(A, -B)*, and *(-A, B)*.

# Example:(Self driving cars )

Self driving cars at an intersection

- ❑ Two cars arrive at the same time at an intersection, and they can decide whether to GO or WAIT.
- ❑ They cannot communicate, and they both want to minimize their waiting time.
- ❑ T>1 is the cost incurred if both cars go at the same time.
- ❑ $v_i$ is the value of the game: if they both wait, they face the exact same game at the next time step

**Are there pure Nash equilibria?**



B

| | | go | wait |
|---|---|---|---|
| A | go | T,T | 0,1 |
| | wait | 1,0 | $1+v_1$,$1+v_2$ |

# Example:(Self driving cars )

Pure Nash equilibria

- Neither(go,go) nor(wait,wait) are NE (think of the no-regret interpretation).
- However, both(go,wait) nor(wait,go) are pure NE.

$$J(go, wait) = (0,1)$$
$$J(wait, go) = (1,0)$$

- Both pure NE are admissible, and they are not interchangeable. It is hard to predict which NE will be played, if any.

| | | B: go | B: wait |
|---|---|---|---|
| A | go | T,T | 0,1 |
| | wait | 1,0 | $1+v_1$,$1+v_2$ |

# Example:(Self driving cars )

## Completely mixed Nash equilibria

Consider the completely mixed strategies

$$y = z = \begin{bmatrix} P(\text{go}) \\ P(wait) \end{bmatrix} = \begin{bmatrix} g \\ 1-g \end{bmatrix} \quad 0 < g < 1$$

B

A

| | go | wait |
|---|---|---|
| go | T,T | 0,1 |
| wait | 1,0 | $1+v_1$,$1+v_2$ |

- ❑ If there is a completely mixed NE strategy, then it must satisfy

$$Az = p1_{2\times 1}$$

- ❑ where p is the value of the game for Player 1 (and Player 2).

$$\begin{bmatrix} T & 0 \\ 1 & 1+p \end{bmatrix} \begin{bmatrix} g \\ 1-g \end{bmatrix} = \begin{bmatrix} p \\ p \end{bmatrix}$$

# Example:(Self driving cars )

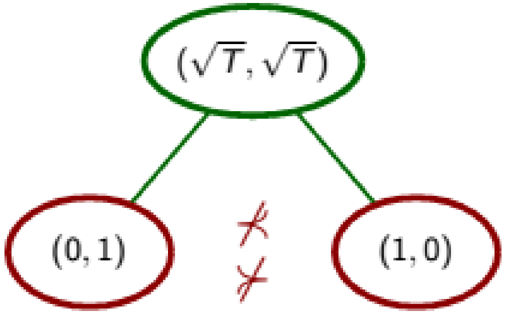
Solving
$$\begin{bmatrix} T & 0 \\ 1 & 1+p \end{bmatrix}\begin{bmatrix} g \\ 1-g \end{bmatrix}=\begin{bmatrix} p \\ p \end{bmatrix}$$

returns the solution
$$g^{*}=\sqrt{\frac{1}{T}}, \quad \text{which satisfies} \quad 0<g^{*}<1$$

There is therefore a unique completely mixed NE, where each car

- ☐ go with probability $\sqrt{1/T}$
- ☐ wait with probability $(1-\sqrt{1/T})$

yielding an outcome $J_1 = J_2 = (\sqrt{T}, \sqrt{T})$



# Computing completely mixed NE

The computation of completely mixed Nash equilibria is particularly simple because, as we saw in the Lemma, all these equilibria must satisfy

$$B^T y^* = q^* 1_{n\times1} \qquad 1^T y^* = 1 \qquad\qquad Az^* = p^* 1_{m\times1} \qquad 1^T z^* = 1$$

which provides a linear system **with n + m + 2** equations and an equal number of unknowns: m entries of y*, n entries of z*, and the two scalars p* and q*.

After solving linear equations, we must still verify that the resulting y* and $z$* do have non-zero entries so that they belong to the sets $Y, Z$ , respectively. It turns out that if they do, we can immediately conclude that we found a Nash equilibria.

# Example:(Battle of sex)

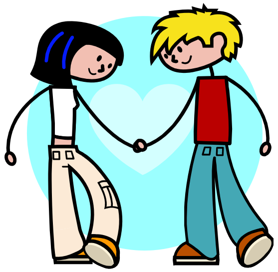
If we consider, instead, the version of the battle of the sexes in previous session , equation now becomes

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$$

So,

$$Az^* = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z^*_1 \\ 1 - z^*_1 \end{bmatrix} = \begin{bmatrix} p^* \\ p^* \end{bmatrix}$$

$$B^T y^* = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} y^*_1 \\ 1 - y^*_1 \end{bmatrix} = \begin{bmatrix} q^* \\ q^* \end{bmatrix}$$

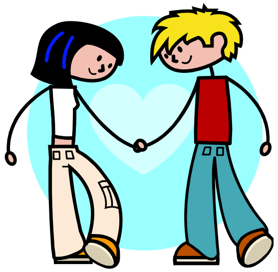


# Example:(Battle of sex)

which is equivalent to

$$-3z^*_1+1=p^*,\ z^*_1-1=p^* \quad \Rightarrow z^*_1=\tfrac{1}{2},\ p^*=-\tfrac{1}{2},\ y^*_1=\tfrac{1}{2},\ q^*=\tfrac{1}{2}$$

$$-3y^*_1+2=q^*, 5y^*_1-2=q^*$$



- ❑ **Nash equilibria**
  - ❑(1,1) is a pure NE with outcome (-2,-1)
  - ❑(2,2) is a pure NE with outcome (−1,-2)
- ❑ This shows that for the version of the game considered in previous session , the only completely mixed Nash equilibria is not admissible, as it is strictly worse for both players than the pure equilibria that we found before.

# Computing mixed NE

We finally consider general mixed NE.

The pair $(y^*, z^*) \in Y \times Z$ is a mixed Nash equilibrium with outcome $(p^*, q^*)$ if and only if $(y^*, z^*, p^*, q^*)$ is the global optimal solution of

$$\min_{y,z,p,q} \; y^T (A+B) z - p - q$$

$$\begin{aligned} \textit{subject to} \quad & Az \geq p 1_{m\times 1} \\ & B^T y \geq q 1_{n\times 1} \\ & z \in Z, y \in Y \end{aligned}$$

Non-convex quadratic program : finding global optimum → hard

# Computing mixed NE

**Proof :**

**Part 1:** we assume that $(y^*, z^*)$ is a mixed Nash equilibrium with outcome equal to $(p^*, q^*)$ and we will show that $(y^*, z^*, p^*, q^*)$ is a global minimum

- **In a NE,** $y^T A z^* \geq y^{*T} A z^* = p^*$ for any $y \in Y$

Therefore also for y that describes the pure strategy i, which selects the i−th element of $Az^*$

- Then, $Az^*$ must be entry-wise greater than $p^*$
- Is a completely similar way, it also follows that $B^T y^*$ must be entry-wise greater than $q^*$

# Computing mixed NE

**Proof :**

- **Part 1:** The $(y^*, z^*, p^*, q^*)$ achieves the global minimum, which is equal to zero. To show this, we first note that since $y^{*T} A z^* = p^*$ and $y^{*T} B z^* = q^*$ we indeed have that

$$y^{*T}(A+B)z^* - p^* - q^* = 0$$

It remains to show that no other vectors y and z that satisfy the constraints can lead to a value for the criteria lower than zero:

$$\gamma^{(i)} = \begin{cases} Az \geq p1 \\ B^T y \geq q1 \end{cases} \Rightarrow \begin{cases} y^T A z \geq p \\ z^T B^T y \geq q \end{cases} \quad \Rightarrow y^T (A+B)z - p - q \geq 0$$

# Computing mixed NE

**Proof :**

**Part 2: if $(y^*, z^*, p^*, q^*)$ is a global minimum, then it is a NE**

As a mixed NE always exists, the argument at the global minimum must be zero.

Moreover, if $Az^* \geq p^* 1_{m\times 1}$, then we must have that $y^T Az^* \geq p^*$ for any $y \in Y$. Similarly, we must have $y^{*T} Bz \geq q^*$ for any $z \in Z$.

Therefore the only way to have a zero argument is to have

$$y^{*T} Az^* = p^* \quad and \quad y^{*T} Bz^* = q^*$$

This means that $(p^*, q^*)$ must be the outcome associated to the mixed strategies $(y^*, z^*)$, and therefore the previous inequalities $y^T Az^* \geq p^*$ *and* $y^{*T} Bz \geq q^*$ imply that $(y^*, z^*)$ is NE.

# MATLAB (Quadratic programs)

[x , val] =quadprog (H , c , Ain , bin , Aeq, beq, low , high , x0)

from MATLAB's Optimization Toolbox numerically solves quadratic programs of the form

$$\min_{x} \ {}^{1}\!/_{2}\, x^{T} H x + C^{T} x$$

$$\begin{aligned} \textit{subject to} \quad & A_{in} x \le b_{in} \\ & A_{eq} x = b_{eq} \\ & low \le x \le high \end{aligned}$$

and returns the value **val** of the minimum and a vector **x** that achieves the minimum.

The **(optional)** vector x0 provides a starting point for the numerical optimization. This is particularly important when **H is** indefinite since in this case the minimization is not convex and may have local **minima.**

# Computing mixed NE

The following MATLAB code can be used to find a mixed Nash equilibrium to the bimatrix game defined by A and B, starting from a random initial condition $x_0$

```
[m,n]=size(A);
x0=rand(n+m+2,1);
% y'(A+b)z-p-q
H=[zeros(m,m),A+B,zeros(m,2);A'+B',zeros(n,n+2);zeros(2,m+n+2)];
c=[zeros(m+n,1);-1;-1];
% A z>=p & B' y >= q
Ain=[zeros(m,m),-A,ones(m,1),zeros(m,1);-B',zeros(n,n+1),ones(n,1)];
bin=zeros(m+n,1);
```

# Computing mixed NE

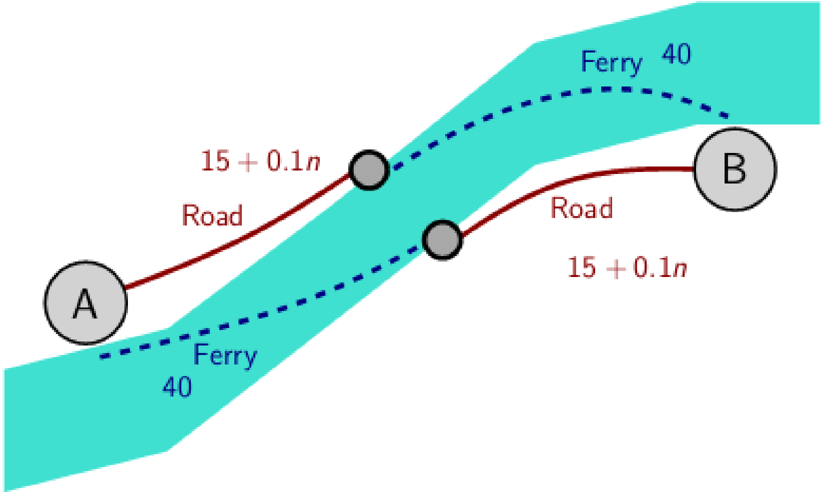
```
% sum(y)=sum(z)=1
Aeq=[ones(1,m),zeros(1,n+2);zeros(1,m),ones(1,n) ,0,0];
beq=[1;1];
% y_i, z_i in [0,1]
low=[zeros(n+m,1);-inf;-inf];
high=[ones(n+m,1);+inf;+inf];
% solve quadratic program
[x,val,exitflag]=quadprog(H,c,Ain,bin,Aeq,beq,low,high,x0)
y=x(1:m)
z=x(m+1:m+n)
p=x(m+n+1)
q=x(m+n+2)
```

# Computing mixed NE

- In this case, the **quadratic** criteria in Theorem is not convex, which means that numerical solvers can easily caught **in local minima.**
- It is therefore important to verify that the solver found a **global minima.**
- This can be easily done since we know that the **global minima** is exactly **equal to zero**.

# Example: Braess paradox

- There are two ways to reach city B from city A, and both include some driving, and a trip on the ferry.
- The two paths are perfectly equivalent, the only difference is whether you first drive, or take the ferry.
- The time needed for the trip depends on what other travelers do.
  - The ferry time is constant, 40 minutes
  - The road time depends on the number of cars on the road.

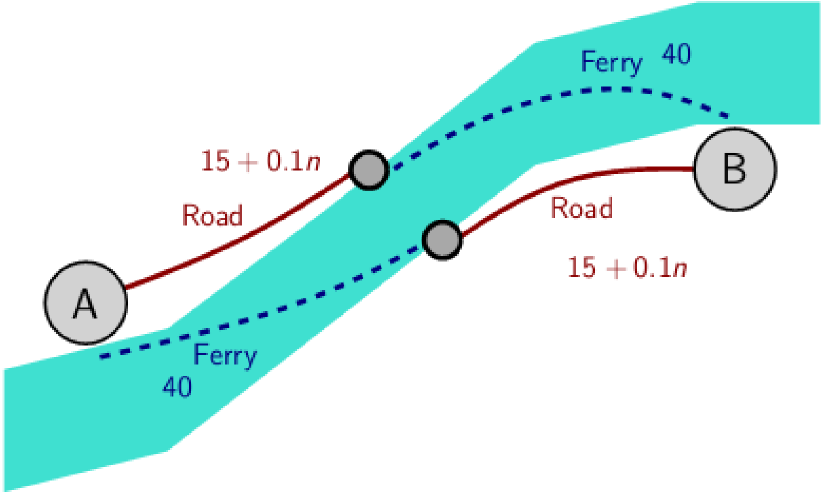


- We consider a population of N=200 travelers.

# Example: Braess paradox

- ❑ Formulation as a non-zero-sum N-person game.
- ❑ Each traveler is a Player.
- ❑ Each Player can decide to take the North or the South path.

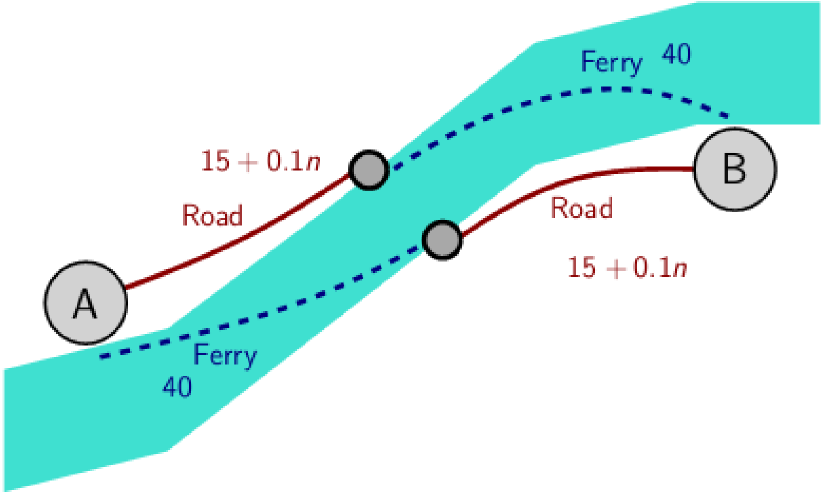
$$\gamma^{(i)} = \begin{cases} 1 & North \\ 0 & South \end{cases}$$



- ❑ All players have identical cost function

$$J_i(\gamma^{(i)},\gamma^{(-i)}) = \begin{cases} 40+15+0.1\sum_j \gamma^{(j)} & if\ \gamma^{(i)}=1 \\ 40+15+0.1\sum_j (1-\gamma^{(\mathrm{j})}) & if\ \gamma^{(i)}=0 \end{cases}$$

# Example: Braess paradox



- ❑ Computing NE is easier when all Players have identical cost function.
- ❑ Are there pure NE?
- ❑ Mixed ones?

$$\gamma^{(i)} = \begin{cases} 1 & with\ P = 0.5 \\ 0 & with\ P = 0.5 \end{cases} \Rightarrow y^{(i)} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

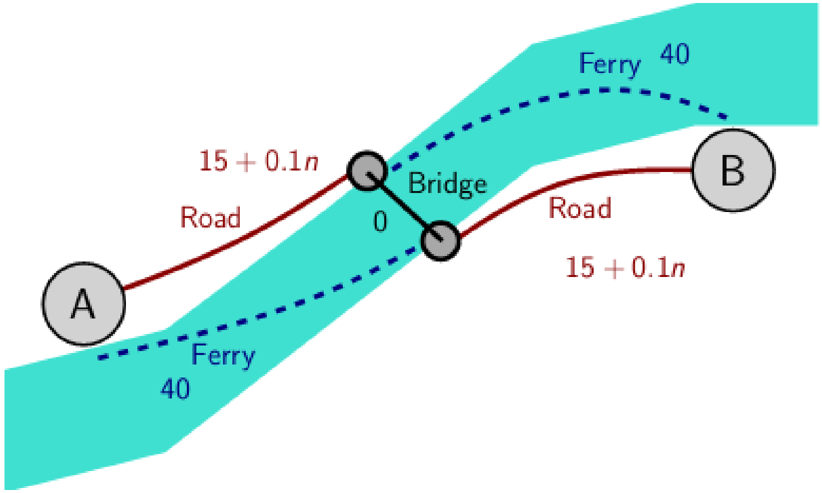
- ❑ In expectation, N/2 travelers will use the North path, and N/2 travelers will use the South path.

$$J_i(y^{(1)*}, \ldots, y^{(N)*}) = 40 + 15 + 0.1 * 200 / 2 = 65 \text{ minutes}$$

- ❑ Can you improve the outcome by unilaterally deviating from the NE?

# Example: Braess paradox

- ❑ Assume a bridge is build, to help reduce traffic.
- ❑ It takes no time to cross the bridge, allowing to go from city A to city B without taking the ferry.

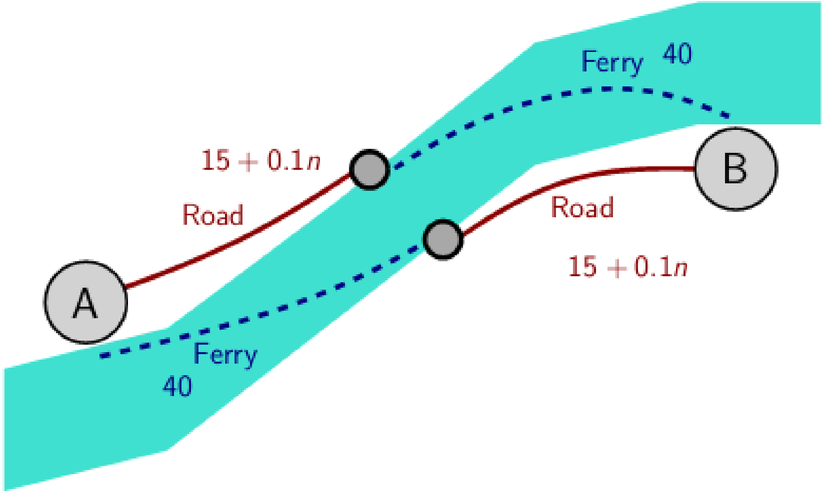


- ❑ New Nash equilibrium: **all travelers avoid the ferry.**

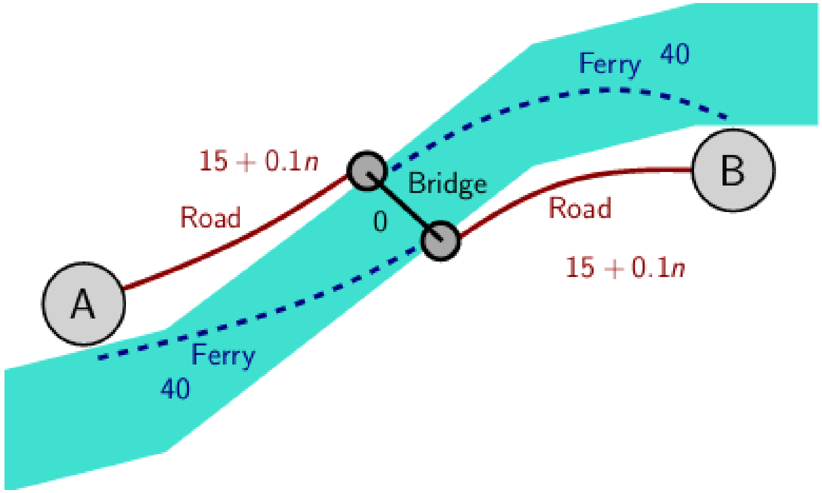
$$J_i(y^{(1)*},\ldots,y^{(N)*}) = 2(15+0.1*200) = 70 \text{ minutes}$$

- ❑ Can you improve the outcome by unilaterally deviating from the NE?
- ❑ No, road + ferry now takes 40+15+0.1·200=75 minutes!

# Example: Braess paradox



$J_i^{NE} = 65$ mins



$J_i^{NE} = 70$ mins

- ❑ With the new link in the transportation graph
  - ❑ the original choice (road + ferry) is still present
  - ❑ the new link is intensively used
  - ❑ all agents experience higher cost!

# Project