کنترل پیش بین
Model Predictive Control

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Optimal Control

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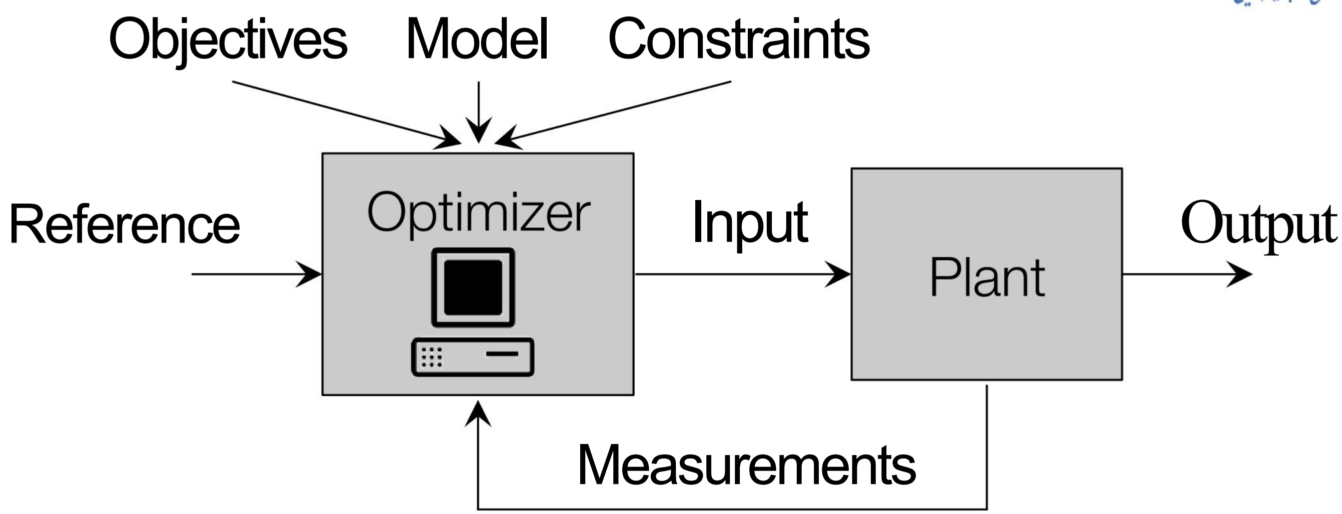
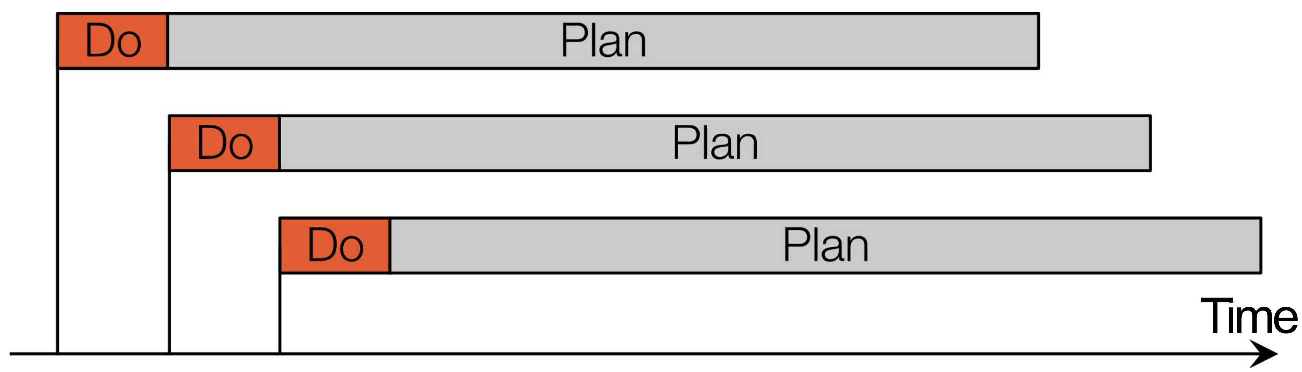
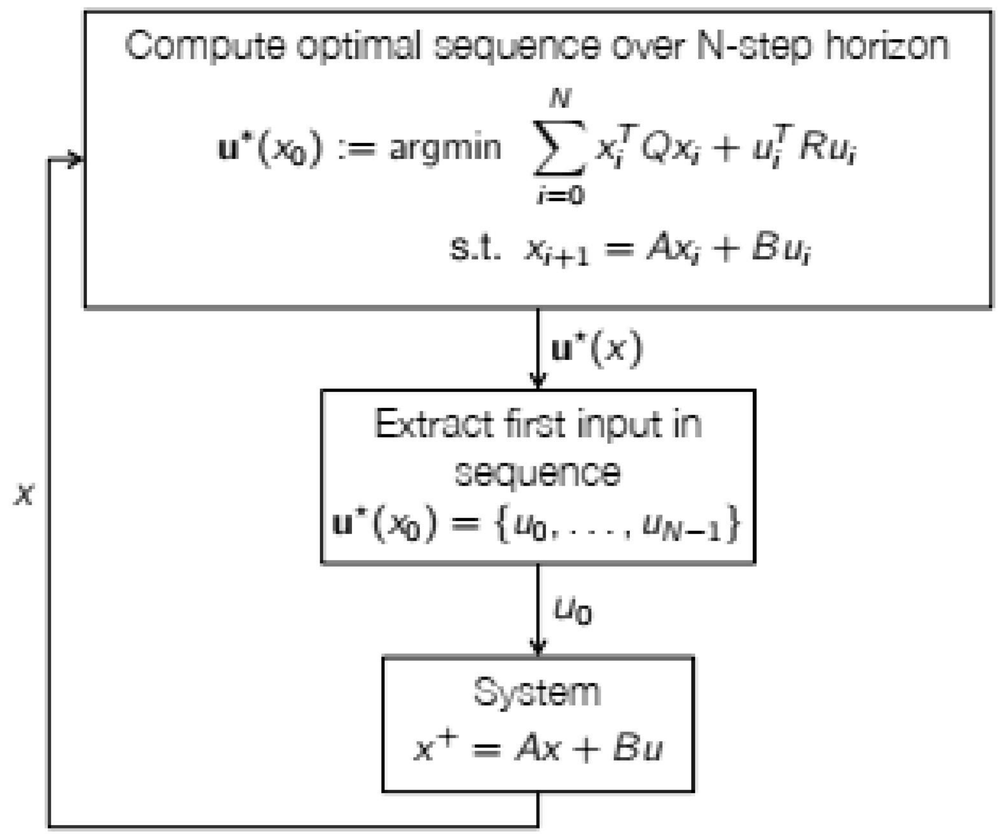
Receding horizon control





Receding horizon strategy introduces feedback.

Receding horizon control



For unconstrained systems, this is a **constant linear controller**

However, can extend this concept to much more complex systems (MPC)

Example - Impact of Horizon Length

Consider the lightly damped, stable system

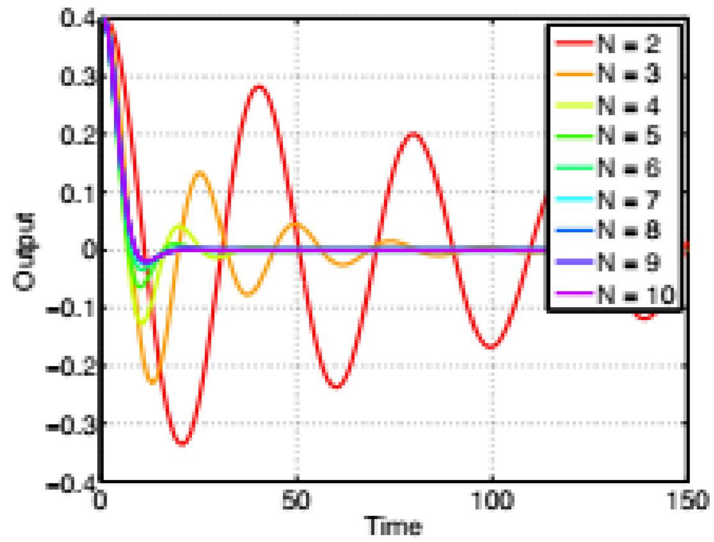
$$G(s) := \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

where $\omega = 1$, $\zeta = 0.01$. We sample at 10Hz and set $P = Q = I$, $R = 1$.

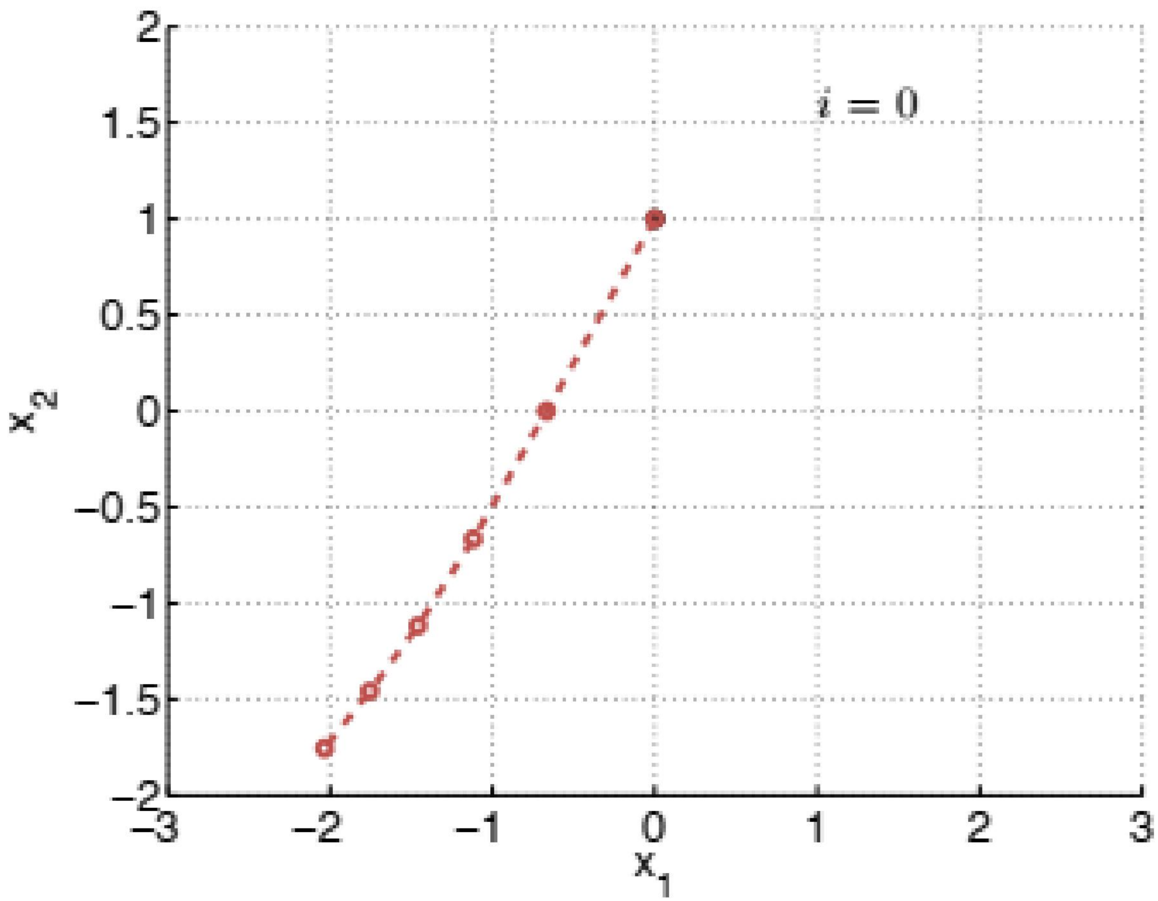
Discrete-time state-space model:

$$x^+ = \begin{bmatrix} 1.988 & -0.998 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0.125 \\ 0 \end{bmatrix} u$$

Closed-loop response

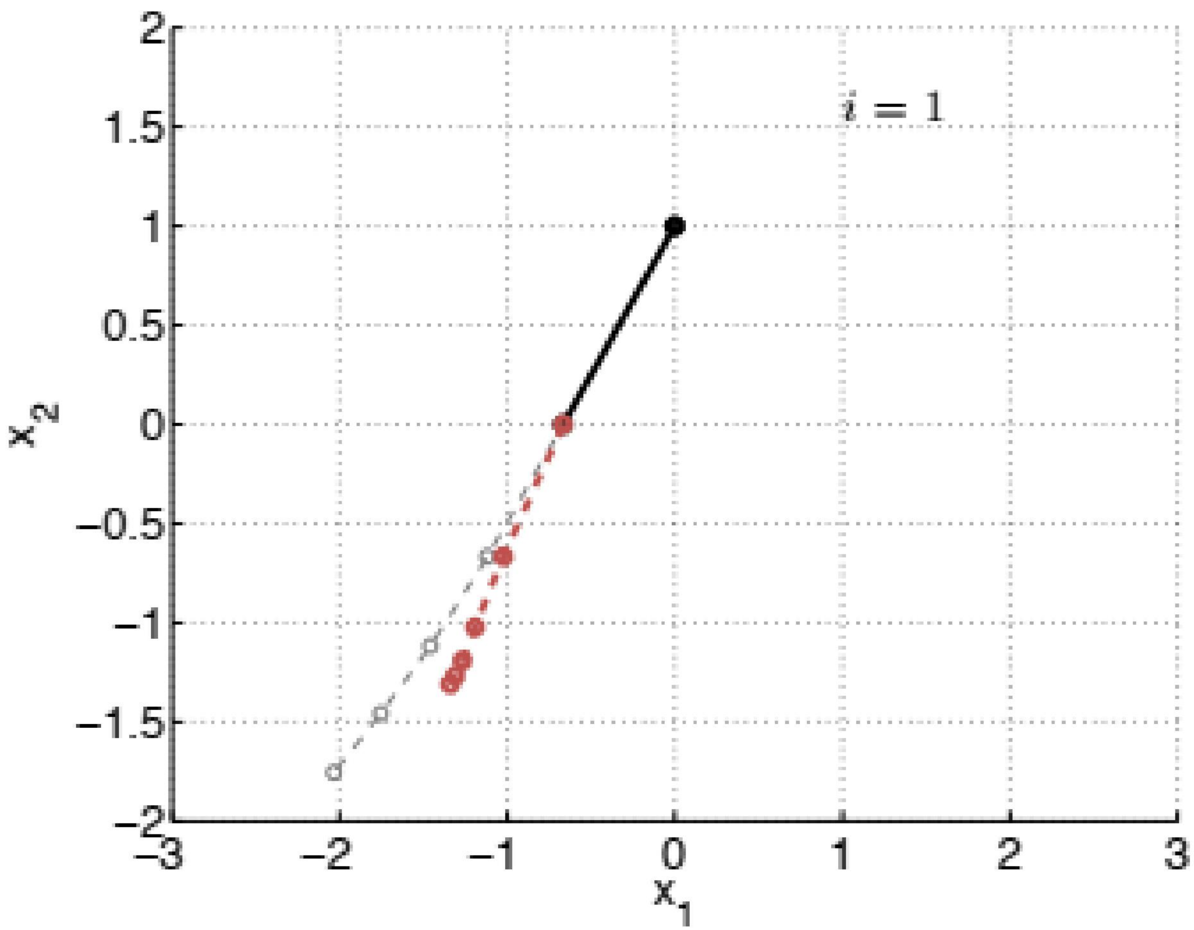


Example: Short horizon N=5



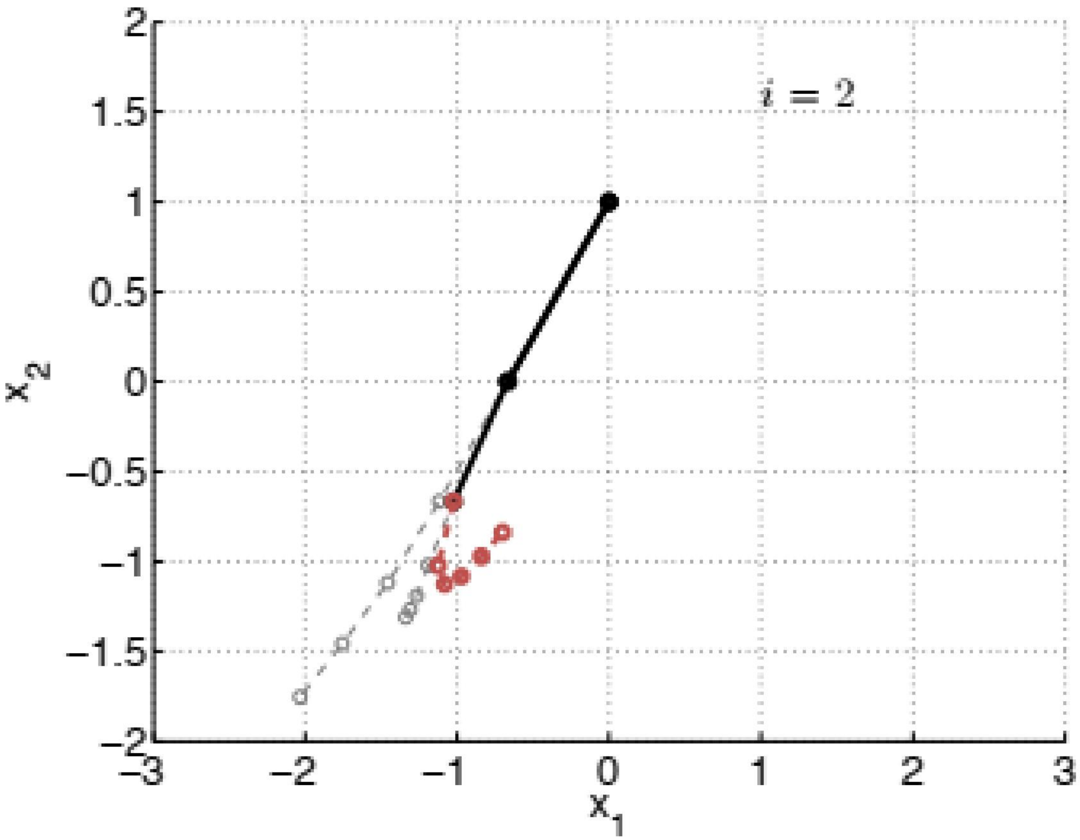
Short horizon: Prediction and closed-loop response differ.

Example: Short horizon N=5



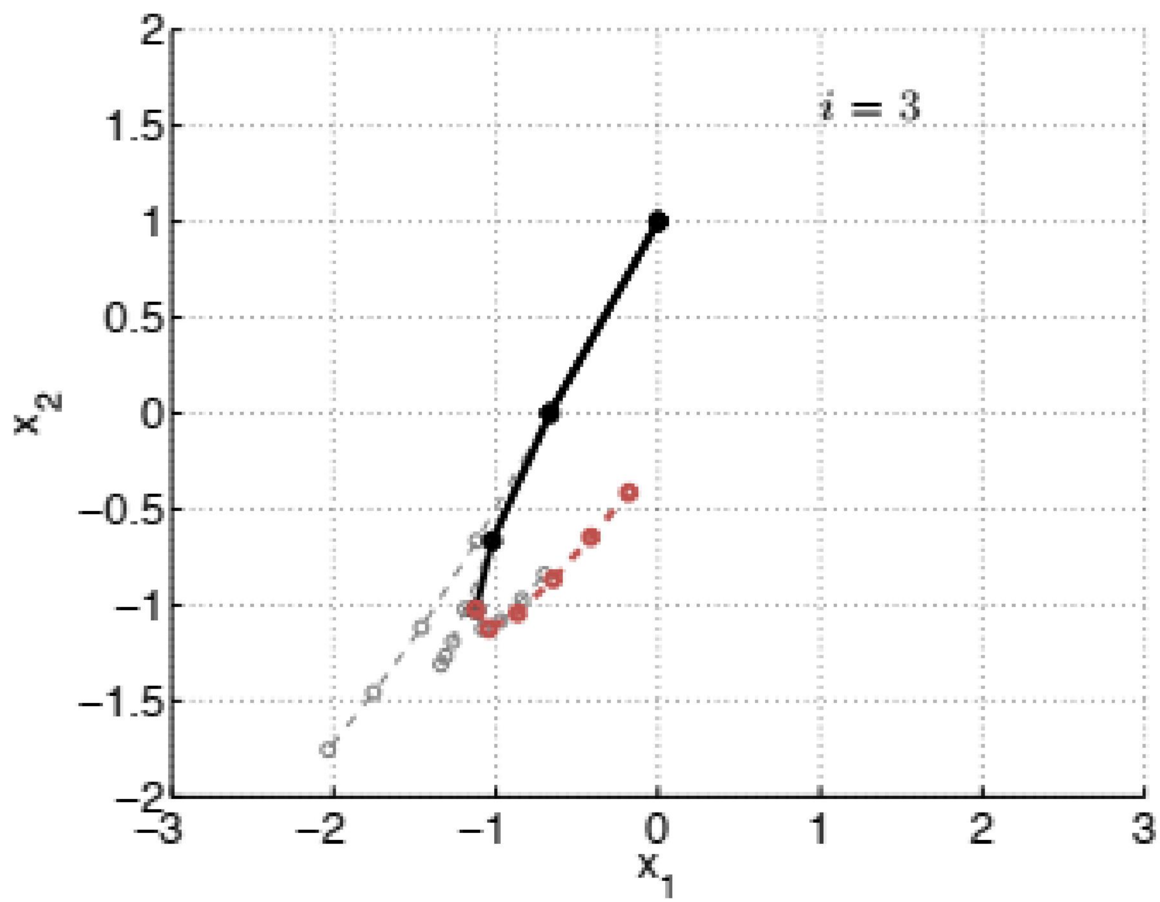
Short horizon: Prediction and closed-loop response differ.

Example: Short horizon N=5



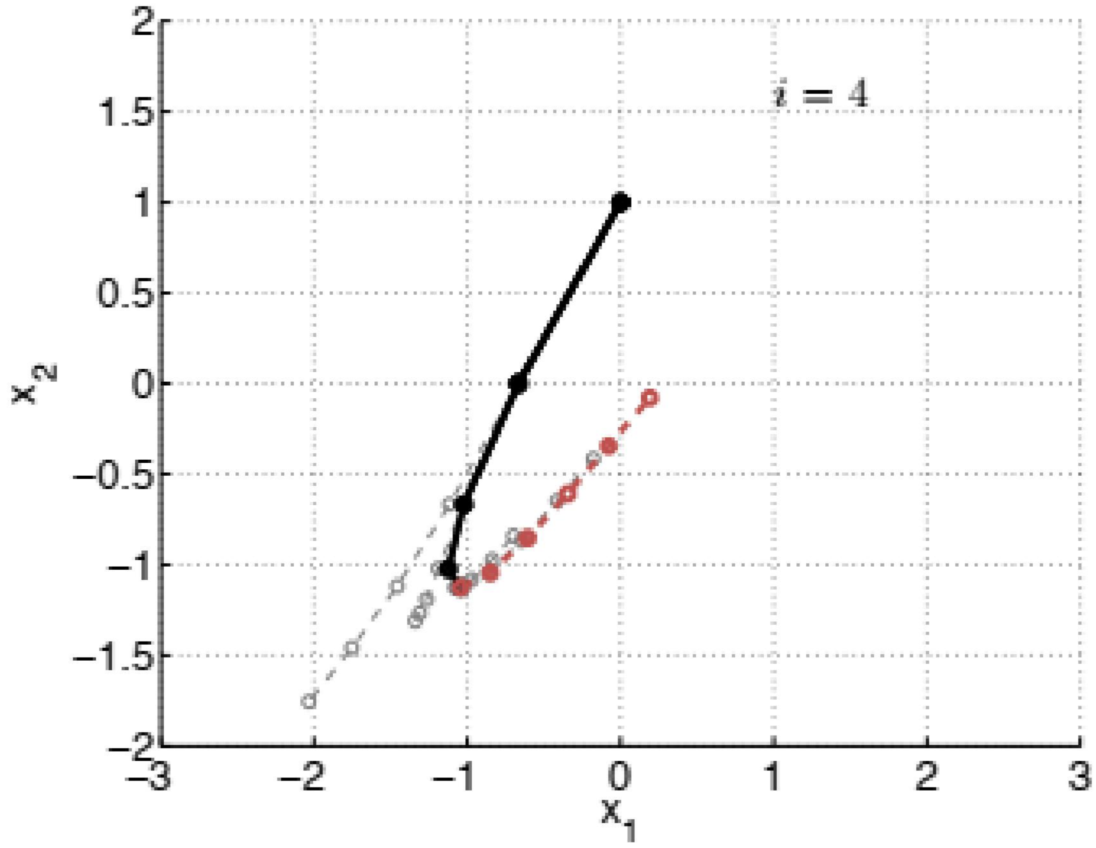
Short horizon: Prediction and closed-loop response differ.

Example: Short horizon N=5



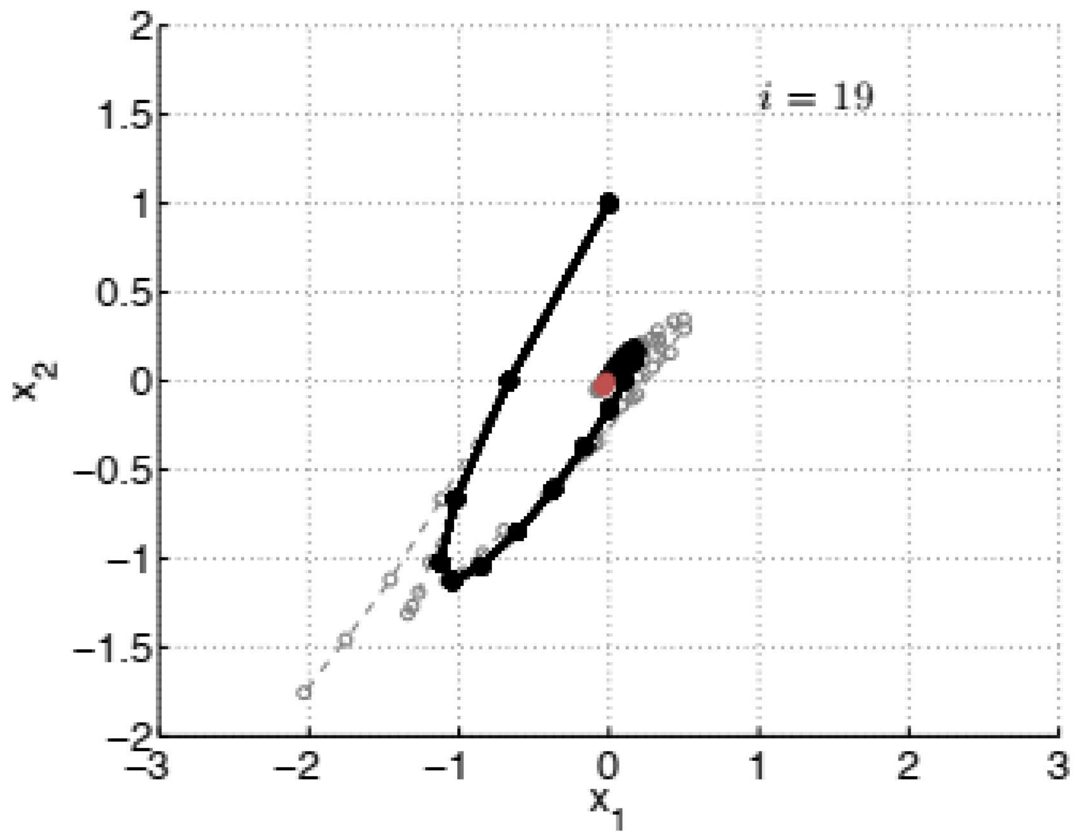
Short horizon: Prediction and closed-loop response differ.

Example: Short horizon N=5



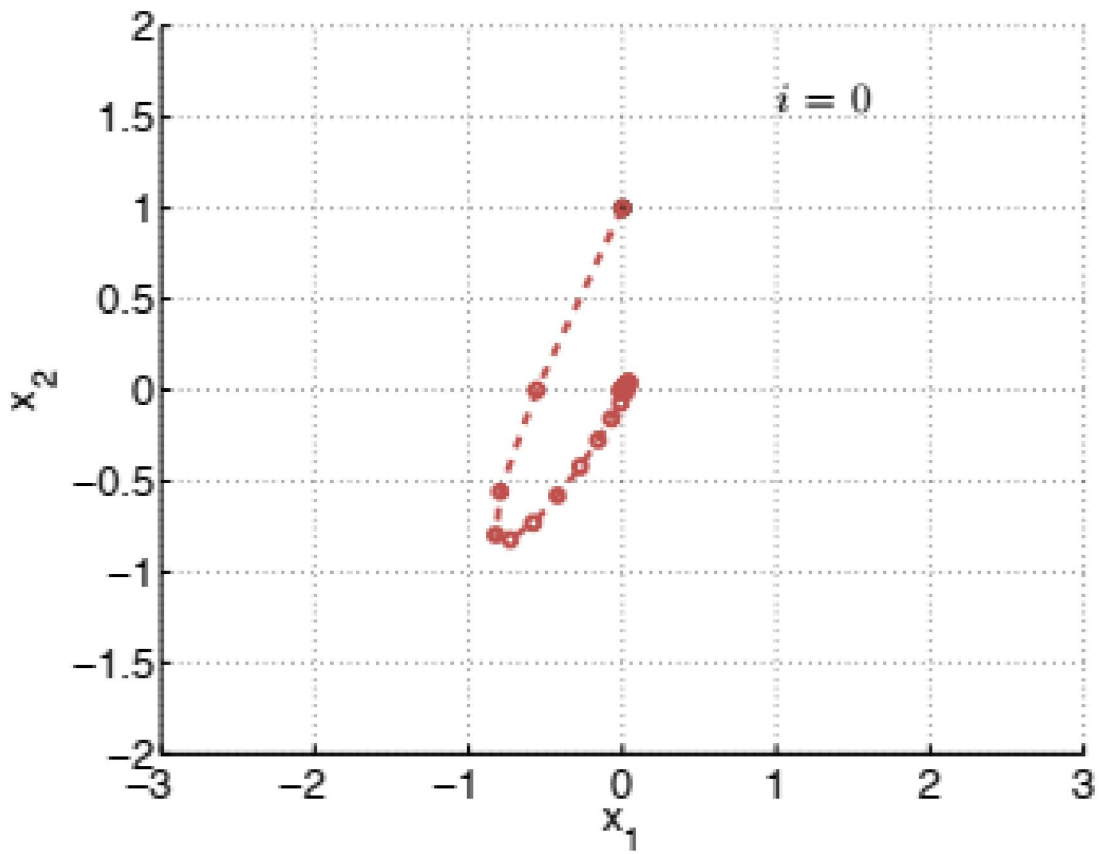
Short horizon: Prediction and closed-loop response differ.

Example: Short horizon N=5



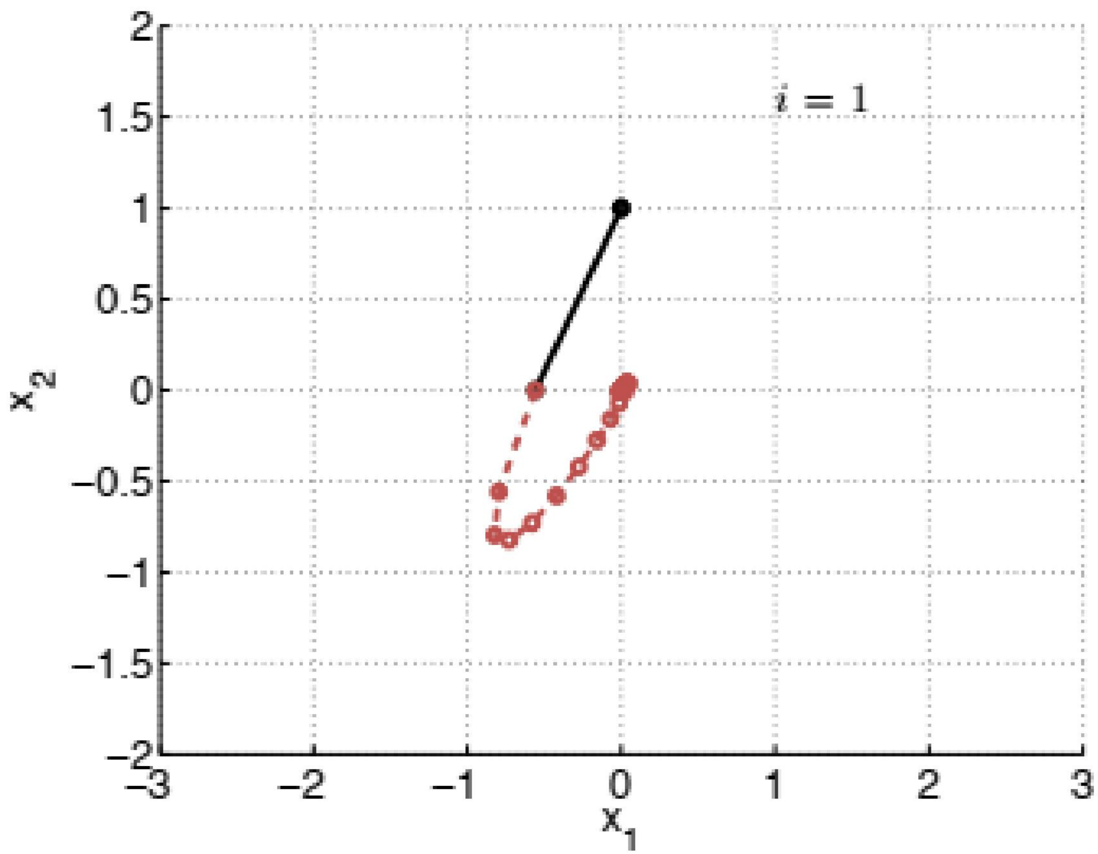
Short horizon: Prediction and closed-loop response differ.

Example: Long horizon N=20



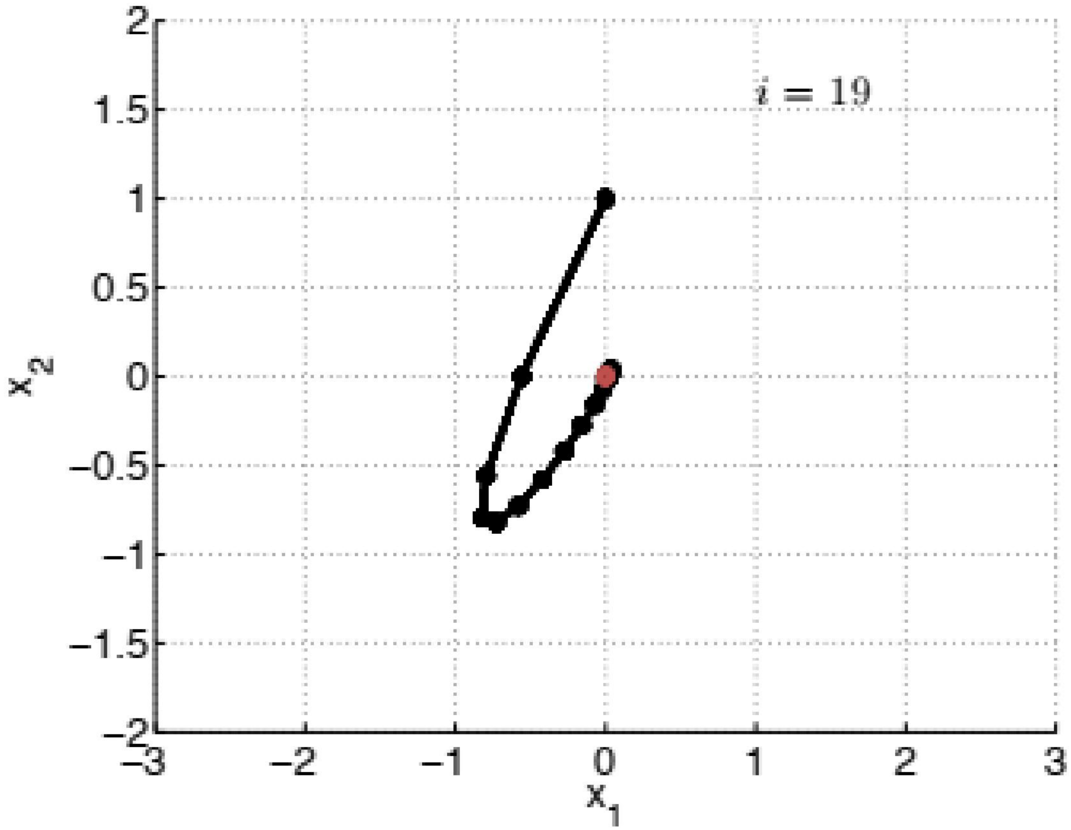
Long horizon: Prediction and closed-loop match.

Example: Long horizon N=20



Long horizon: Prediction and closed-loop match.

Example: Long horizon N=20



Long horizon: Prediction and closed-loop match.

Linear Quadratic Optimal Control

Infinite Horizon Control Problem: Optimal Solution

- In some cases we may want to solve the same problem with an infinite horizon:

$$J_\infty(x(0)) = \min_{u(\cdot)} \left\{ \sum_{k=0}^{\infty} [x_k' Q x_k + u_k' R u_k] \right\}$$

$$\text{subject to} \quad x_{k+1} = A x_k + B u_k, \quad k = 0, 1, 2, \ldots, \infty,$$
$$x_0 = x(0)$$

- As with the Dynamic Programming approach, the optimal input is of the form.

$$u^*(k) = -(B' P_\infty B + R)^{-1} B' P_\infty A x(k) \triangleq F_\infty x(k)$$

and the infinite-horizon cost-to-go is.

$$J_\infty(x(k)) = x(k)' P_\infty x(k) .$$

Linear Quadratic Optimal Control

Infinite Horizon Control Problem: Optimal Solution

- The matrix P_∞ comes from an infinite recursion of the RDE, from a point infinitely far into the future.
- Assuming the RDE does converge to some constant matrix P_∞, it must satisfy the following (from (7), with $P_k = P_{k+1} = P_\infty$)

$$P_\infty = A' P_\infty A + Q - A' P_\infty B (B' P_\infty B + R)^{-1} B' P_\infty A \,,$$

which is called the Algebraic Riccati Equation (ARE).

- The constant feedback matrix F_∞ is referred to as the asymptotic form of the Linear Quadratic Regulator (LQR).
- In fact, if (A, B) is controllable and (Q, A) is observable, then the RDE (initialized with Q at $k = \infty$ and solved for $k \searrow 0$) converges to the unique positive definite solution P_∞ of the ARE

Linear Quadratic Optimal Control

Stability of Infinite-Horizon LQR

- In addition, the closed-loop system with $u(k) = F_\infty x(k)$ is guaranteed to be asymptotically stable, under the stabilizability and detectability assumptions of the previous slide.
- The latter statement can be proven by substituting the control law $u(k) = F_\infty x(k)$ into $x(k + 1) = A\, x(k) + B\, u(k)$, and then examining the properties of the system

$$x(k+1) = (A + BF_\infty)x(k)\,. \tag{8}$$

The asymptotic stability of (8) can be proven by showing that the infinite horizon cost $J^*_\infty(x(k)) = x(k)' P_\infty x(k)$ is actually a Lyapunov function for the system, i.e. $J^*_\infty(x(k)) > 0,\ \forall k \neq 0,\ J^*_\infty(0) = 0$, and $J^*_\infty(x(k+1)) < J^*_\infty(x(k))$, for any $x(k)$. This implies that

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Constrained Linear Optimal Control

Cost function

$$J_0(x(0), U_0) = p(x_N) + \sum_{k=0}^{N-1} q(x_k, u_k)$$

- $U_0 \triangleq [u_0', \ldots, u_{N-1}']'$
- Squared Euclidian norm: $p(x_N) = x_N' P x_N$ and $q(x_k, u_k) = x_k' Q x_k + u_k' R u_k$.
- $p = 1$ or $p = \infty$: $p(x_N) = \|Px_N\|_p$ and $q(x_k, u_k) = \|Qx_k\|_p + \|Ru_k\|_p$.

Constrained Finite Time Optimal Control problem (CFTOC)

$$\begin{array}{lll} J_0^*(x(0)) = & \min_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \tag{9}$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Constrained Linear Optimal Control

Feasible Sets

Set of initial states $x(0)$ for which the optimal control problem (9) is feasible:

$$\begin{aligned}\mathcal{X}_0 = \; & \{x_0 \in \mathbb{R}^n | \; \exists (u_0, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X}, \; u_k \in \mathcal{U}, \\ & k = 0, \ldots, N-1, \; x_N \in \mathcal{X}_f, \text{ where } x_{k+1} = Ax_k + Bu_k\}\end{aligned}$$

In general $\mathcal{X}_i$ is the set of states x_i at time i for which (9) is feasible:

$$\begin{aligned}\mathcal{X}_i = \; & \{x_i \in \mathbb{R}^n | \; \exists (u_i, \ldots, u_{N-1}) \text{ such that } x_k \in \mathcal{X}, \; u_k \in \mathcal{U}, \\ & k = i, \ldots, N-1, \; x_N \in \mathcal{X}_f, \text{ where } x_{k+1} = Ax_k + Bu_k\},\end{aligned}$$

The sets $\mathcal{X}_i$ for $i = 0, \ldots, N$ play an important role in the the solution of the CFTOC problem. They are independent of the cost.

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Constrained Optimal Control: 2-Norm

Problem Formulation

Quadratic cost function

$$J_0(x(0), U_0) = x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k \qquad (10)$$

with $P \succeq 0$, $Q \succeq 0$, $R \succ 0$.
Constrained Finite Time Optimal Control problem (CFTOC).

$$\begin{aligned} J_0^*(x(0)) = \quad & \min_{U_0} && J_0(x(0), U_0) \\ & \text{subj. to} && x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & && x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & && x_N \in \mathcal{X}_f \\ & && x_0 = x(0) \end{aligned} \qquad (11)$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Constrained Optimal Control: 2-Norm

Construction of the QP with substitution

- **Step 1**: Rewrite the cost as (see previous slides)

$$
\begin{aligned}
J_0(x(0), U_0) &= U_0' H U_0 + 2x(0)' F U_0 + x(0)' Y x(0) \\
&= [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0' \; x(0)']'
\end{aligned}
$$

- **Step 2**: Rewrite the constraints compactly as (details provided on the next slide)

$$G_0 U_0 \leq w_0 + E_0 x(0)$$

- **Step 3**: Rewrite the optimal control problem as

$$
\begin{aligned}
J_0^*(x(0)) = \min_{U_0} \quad & [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0' \; x(0)']' \\
\text{subj. to} \quad & G_0 U_0 \leq w_0 + E_0 x(0)
\end{aligned}
$$

Constrained Optimal Control: 2-Norm

Construction of the QP with substitution

Solution

$$J_0^*(x(0)) = \min_{U_0} \quad [U_0' \ x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Y \end{smallmatrix}\right] [U_0' \ x(0)']'$$
$$\text{subj. to} \quad G_0 U_0 \leq w_0 + E_0 x(0)$$

For a given x(0) U_0^* can be found via a QP solver.

Constrained Optimal Control: 2-Norm

Construction of the QP with substitution

If $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ are given by:

$$\mathcal{X} = \{x \,|\, A_x x \leq b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \leq b_u\} \qquad \mathcal{X}_f = \{x \,|\, A_f x \leq b_f\}$$

Then G_0, E_0 and w_0 are defined as follows

$$G_0 = \begin{bmatrix} A_u & 0 & \dots & 0 \\ 0 & A_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_u \\ 0 & 0 & \dots & 0 \\ A_x B & 0 & \dots & 0 \\ A_x A B & A_x B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_f A^{N-1} B & A_f A^{N-2} B & \dots & A_f B \end{bmatrix}, E_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -A_x \\ -A_x A \\ -A_x A^2 \\ \vdots \\ -A_f A^N \end{bmatrix}, w_0 = \begin{bmatrix} b_u \\ b_u \\ \vdots \\ b_u \\ b_x \\ b_x \\ b_x \\ \vdots \\ b_f \end{bmatrix}$$

Constrained Optimal Control: 2-Norm

Construction of the QP without substitution

To obtain the QP problem

$$J_0^*(x(0)) = \min_{U_0} \quad [U_0' \; x(0)'] \left[\begin{smallmatrix} H & F' \\ F & Q \end{smallmatrix} \right] [U_0' \; x(0)']'$$
$$\text{subj. to} \quad G_0 U_0 \leq w_0 + E_0 x(0)$$

we have substituted the state equations

$$x_{k+1} = A x_k + B u_k$$

into the state constraints $x_k \in \mathcal{X}$.

It is often more efficient to keep the explicit equality constraints.

Constrained Optimal Control: 2-Norm

Construction of the QP without substitution

We transform the CFTOC problem into the QP problem

$$
\begin{aligned}
J_0^*(x(0)) = \min_{z} \quad & [z' \;\; x(0)'] \begin{bmatrix} \bar{H} & 0 \\ 0 & Q \end{bmatrix} [z' \;\; x(0)']' \\
\text{subj. to} \quad & G_{0,\text{in}} z \le w_{0,\text{in}} + E_{0,\text{in}} x(0) \\
& G_{0,\text{eq}} z = E_{0,\text{eq}} x(0)
\end{aligned}
$$

- Define variable:

$$z = \begin{bmatrix} x_1' & \ldots & x_N' & u_0' & \ldots & u_{N-1}' \end{bmatrix}'$$

- Equalities from system dynamics $x_{k+1} = Ax_k + Bu_k$:

$$
G_{0,\text{eq}} = \left[\begin{array}{ccccc|ccccc}
I & & & & & -B & & & & \\
-A & I & & & & & -B & & & \\
 & -A & I & & & & & -B & & \\
 & & \ddots & \ddots & & & & & \ddots & \\
 & & & -A & I & & & & & -B
\end{array}\right], \quad E_{0,\text{eq}} = \begin{bmatrix} A \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Constrained Optimal Control: 2-Norm

Construction of the QP without substitution

If $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{X}_f$ are given by:

$$\mathcal{X} = \{x \,|\, A_x x \leq b_x\} \qquad \mathcal{U} = \{u \,|\, A_u u \leq b_u\} \qquad \mathcal{X}_f = \{x \,|\, A_f x \leq b_f\}$$

Then matrices $G_{0,\text{in}}$, $w_{0,\text{in}}$ and $E_{0,\text{in}}$ are:

$$G_{0,\text{in}} = \left[\begin{array}{ccccc|ccccc} 0 & & & & & 0 & & & & \\ & A_x & & & & & 0 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & A_x & & & & & 0 & \\ & & & & A_f & & & & & 0 \\ \hline 0 & & & & & A_u & & & & \\ & 0 & & & & & A_u & & & \\ & & \ddots & & & & & \ddots & & \\ & & & 0 & & & & & A_u & \\ & & & & 0 & & & & & A_u \end{array}\right] \qquad w_{0,\text{in}} = \left[\begin{array}{c} b_x \\ b_x \\ \vdots \\ b_x \\ b_f \\ \hline b_u \\ b_u \\ \vdots \\ b_u \\ b_u \end{array}\right]$$

$$E_{0,\text{in}} = \begin{bmatrix} -A_x' & 0 & \cdots & 0 \end{bmatrix}'$$

Constrained Optimal Control: 2-Norm

Construction of the QP without substitution

Build cost function from MPC cost $x_N' P x_N + \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k$

$$\bar{H} = \left[\begin{array}{cccc|ccc} Q & & & & & & \\ & \ddots & & & & & \\ & & Q & & & & \\ & & & P & & & \\ \hline & & & & R & & \\ & & & & & \ddots & \\ & & & & & & R \end{array}\right]$$

Matlab hint:

```
barH = blkdiag(kron(eye(N-1),Q), P, kron(eye(N),R))
```

Constrained Optimal Control: 2-Norm

2-Norm State Feedback Solution

Start from QP with substitution.

- **Step 1**: Define $z \triangleq U_0 + H^{-1}F'x(0)$ and transform the problem into

$$\hat{J}^*(x(0)) = \min_z \; z'Hz \quad \text{subj. to} \quad G_0 z \le w_0 + S_0 x(0),$$

where $S_0 \triangleq E_0 + G_0 H^{-1}F'$, and
$\hat{J}^*(x(0)) = J_0^*(x(0)) - x(0)'(Y - FH^{-1}F')x(0)$.

The CFTOC problem is now a **multiparametric quadratic program (mp-QP)**.

- **Step 2**: Solve the mp-QP to get explicit solution $z^*(x(0))$

- **Step 3**: Obtain $U_0^*(x(0))$ from $z^*(x(0))$

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Constrained Optimal Control: 2-Norm

Problem Formulation

Piece-wise linear cost function

$$J_0(x(0), U_0) := \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \|Ru_k\|_p \qquad (12)$$

with $p = 1$ or $p = \infty$, P, Q, R full column rank matrices

Constrained Finite Time Optimal Control Problem (CFTOC)

$$\begin{array}{lll} J_0^*(x(0)) = & \min\limits_{U_0} & J_0(x(0), U_0) \\ & \text{subj. to} & x_{k+1} = Ax_k + Bu_k,\ k = 0, \ldots, N-1 \\ & & x_k \in \mathcal{X},\ u_k \in \mathcal{U},\ k = 0, \ldots, N-1 \\ & & x_N \in \mathcal{X}_f \\ & & x_0 = x(0) \end{array} \qquad (13)$$

N is the time horizon and $\mathcal{X}$, $\mathcal{U}$, $\mathcal{X}_f$ are polyhedral regions.

Constrained Optimal Control: 2-Norm

Construction of the LP with substitution

The problem results in the following standard LP

$$\begin{array}{ll} \min\limits_{z_0} & c_0' z_0 \\ \text{subj. to} & \bar{G}_0 z_0 \leq \bar{w}_0 + \bar{S}_0 x(0) \end{array}$$

where $z_0 := \{\varepsilon_0^x, \ldots, \varepsilon_N^x, \varepsilon_0^u, \ldots, \varepsilon_{N-1}^u, u_0', \ldots, u_{N-1}'\} \in \mathbb{R}^s$, $s \triangleq (m+1)N + N + 1$ and

$$\bar{G}_0 = \begin{bmatrix} G_\varepsilon & 0 \\ 0 & G_0 \end{bmatrix}, \quad \bar{S}_0 = \begin{bmatrix} S_\varepsilon \\ S_0 \end{bmatrix}, \quad \bar{w}_0 = \begin{bmatrix} w_\varepsilon \\ w_0 \end{bmatrix}$$

For a given $x(0)$ U_0^* can be obtained via an LP solver (the $1-$norm case is similar).

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