کنترل پیش بین
Model Predictive Control

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Robust MPC

- Uncertainty Models
- Impact of Bounded Additive Noise
- Robust Open-Loop MPC
- Closed-Loop Predictions
- Tube-MPC

Reference:

F. Borrelli, A. Bemporad, and M. Morari, Predictive Control for Linear and Hybrid Systems, Cambridge University Press, 2017. [Ch. 12].

Uncertainty Models

- MPC relies on a model, but models are far from perfect
- Noise and model inaccuracies can cause:
  - Constraint violation
  - Sub-optimal behaviour can result
- Persistent noise prevents the system from converging to a single point
- Can incorporate some noise models into the MPC formulation
  - Solving the resulting optimal control problem is extremely difficult
  - Many approximations exist, but most are very conservative

Examples of Common Uncertainty Models

Additive Bounded Noise

$$g(x, u, w; \theta) = Ax + Bu + w \ , \quad w \in \mathbb{W}$$

A, B known, w unknown and changing with each sample

- Dynamics are linear, but impacted by random, bounded noise at each time step
- Can model many nonlinearities in this fashion, but often a conservative model
- The noise is persistent, i.e., it does not converge to zero in the limit

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Goals of Robust Constrained Control

Uncertain constrained linear system
$x^+ = Ax + Bu + w \qquad (x,u) \in \mathcal{X}, \mathcal{U} \qquad w \in \mathbb{W}$

- Design control law u = K(x) such that the system:

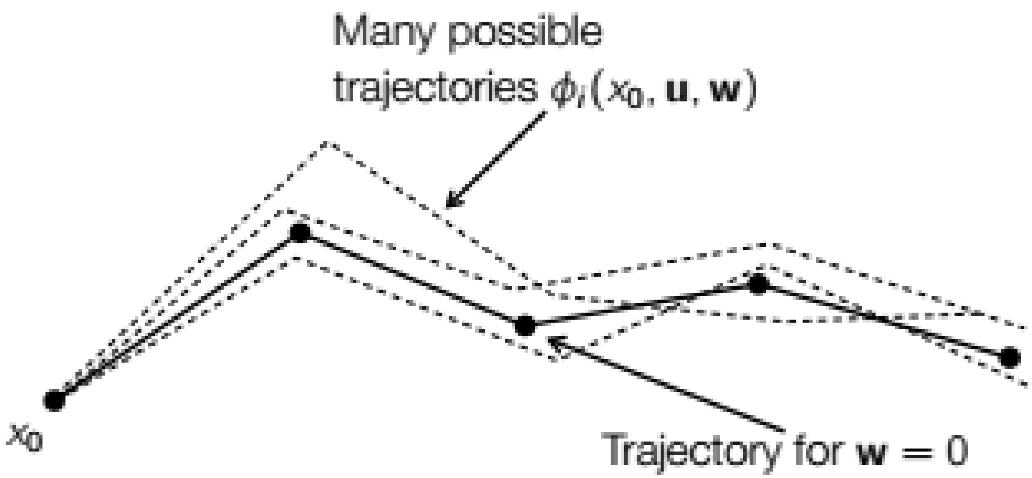
1. Satisfies constraints : $\{x_i\} \dashrightarrow X$, $\{u_i\} \dashrightarrow U$ for all disturbance realizations
2. Is stable: Converges to a neighborhood of the origin
3. Optimizes (expected/worst-case) "performance"
4. Maximizes the set {x0 | Conditions 1-3 are met}

Challenge: Cannot predict where the state of the system will evolve
We can only compute a set of trajectories that the system *may* follow

Idea: Design a control law that will satisfy constraints and stabilize the system for all possible disturbances

Uncertain State Evolution

Given the current state x_0, the model $x^+ = Ax + Bu + w$ and the set $\mathbb{W}$, where can the state be i steps in the future?



Define $\phi_i(x_0, \vec{u}, \vec{w})$ as the state that the system will be in at time i if the state at time zero is x_0, we apply the input $\vec{u} := \{u_0, \ldots, u_{N-1}\}$ and we observe the disturbance $\vec{w} := \{w_0, \ldots, w_{N-1}\}$.

Uncertain State Evolution

Nominal system

$$x^+ = Ax + Bu$$

$$x_1 = Ax_0 + Bu_0$$
$$x_2 = A^2x_0 + ABu_0 + Bu_1$$
$$\vdots$$
$$x_i = A^i x_0 + \sum_{k=0}^{i-1} A^k Bu_{i-k}$$

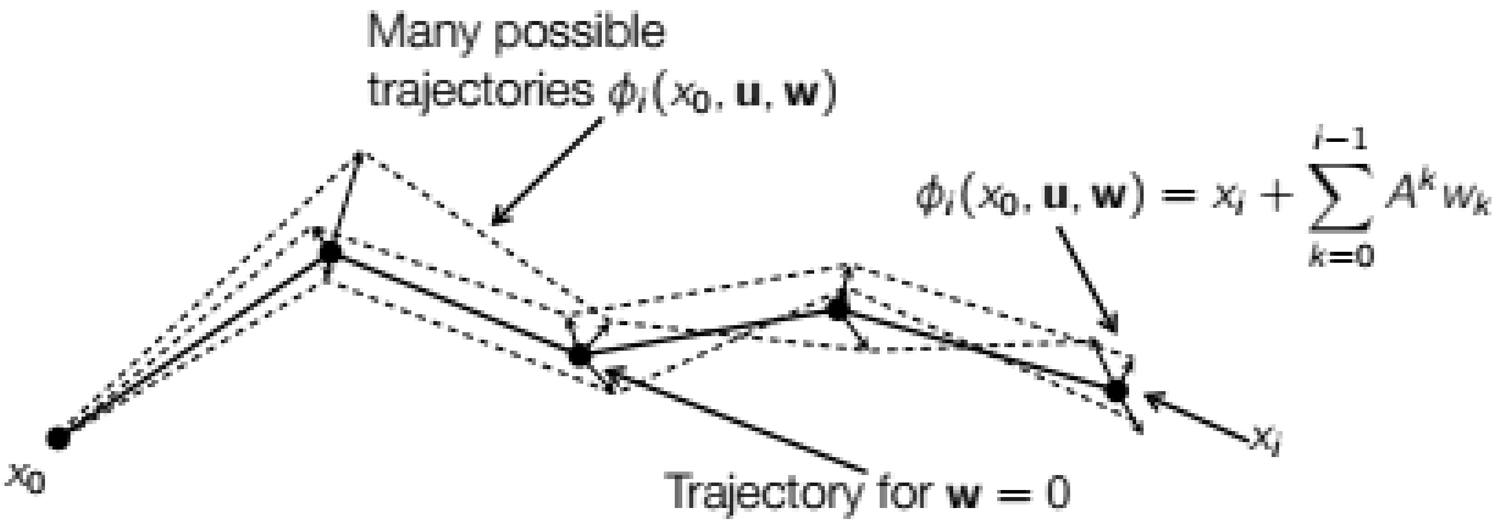
Uncertain system

$$x^+ = Ax + Bu + w \ , w \in \mathbb{W}$$

$$\phi_1 = Ax_0 + Bu_0 + w_0$$
$$\phi_2 = A^2x_0 + ABu_0 + Bu_1 + Aw_0 + w_1$$
$$\vdots$$
$$\phi_i = A^i x_0 + \sum_{k=0}^{i-1} A^k Bu_{i-k} + \sum_{k=0}^{i-1} A^k w_{i-k}$$
$$\phi_i = x_i + \sum_{k=0}^{i-1} A^k w_{i-k}$$

Uncertain evolution is the nominal system + offset caused by the disturbance
(Follows from linearity)

Uncertain State Evolution



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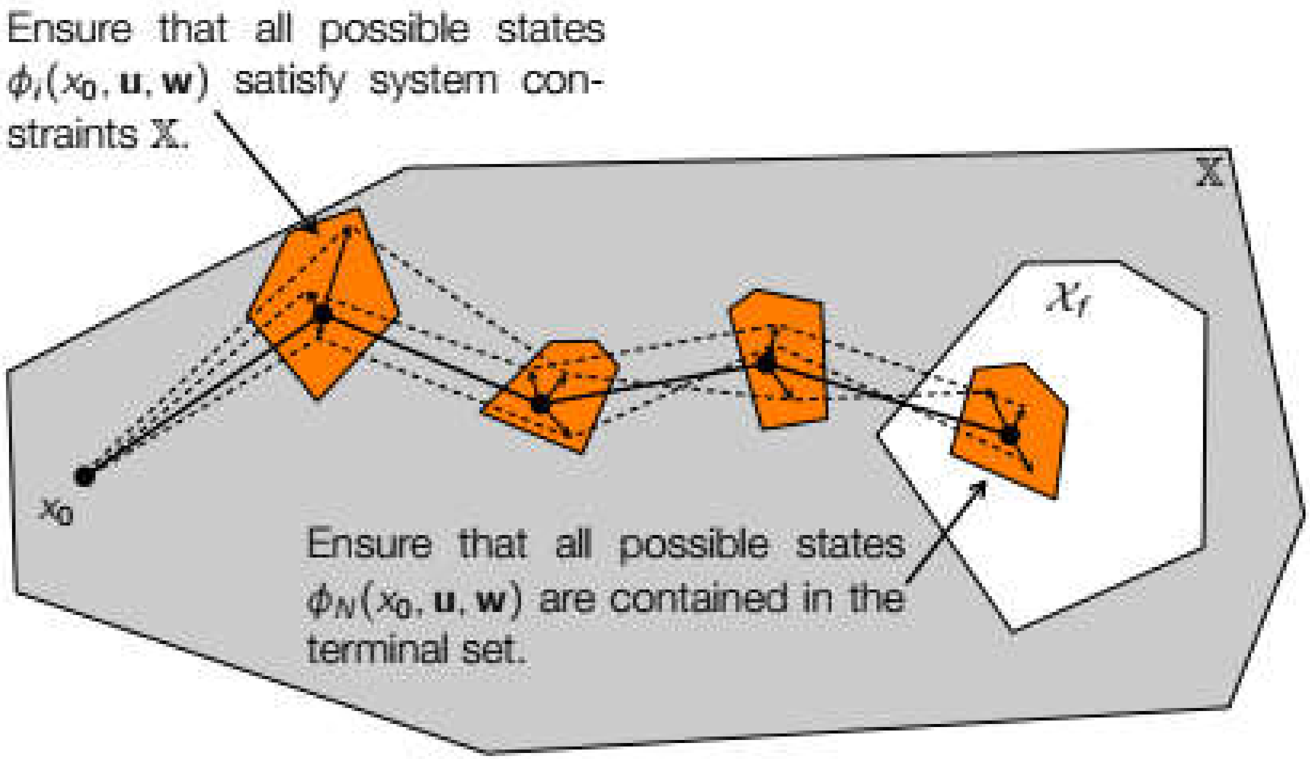
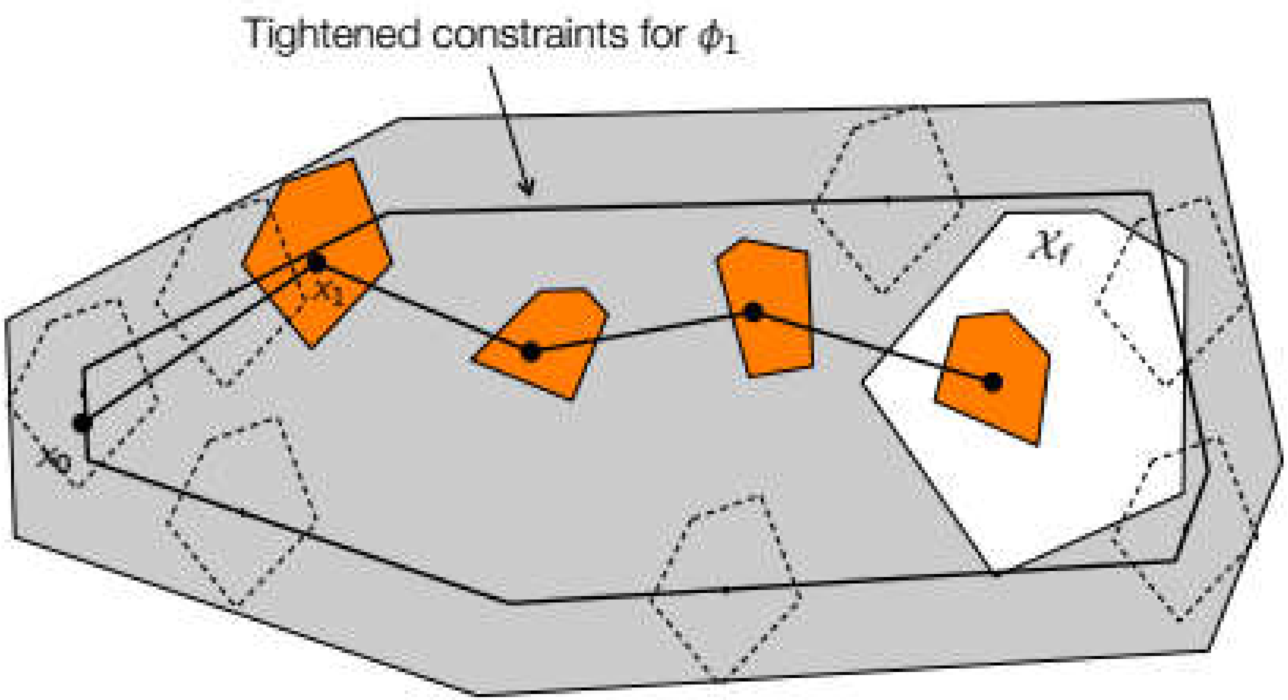
Robust Constraint Satisfaction



The idea: Compute a set of tighter constraints such that if **the nominal system** meets these constraints, then the uncertain system will too.
We then do MPC **on the nominal system.**

Robust Constraint Satisfaction

Goal: Ensure that constraints are satisfied for the MPC sequence.



Require: $x_i \in \mathcal{X} \ominus \begin{bmatrix} I & A^0 & \dots & A^{i-1} \end{bmatrix} \mathbb{W}^i$ and

Nominal x_i satisfies tighter constraints → Uncertain state does too

Putting it Together

Robust Open-Loop MPC

$$\min_{\vec{u}} \sum_{i=0}^{N-1} l(x_i, u_i) + V_f(x_N)$$

$$\begin{aligned} \text{subj. to } x_{i+1} &= Ax_i + Bu_i \\ x_i &\in \mathcal{X} \ominus \mathcal{A}_i \mathbb{W}^i \\ u_i &\in \mathcal{U} \\ x_N &\in \tilde{\mathcal{X}}_f \end{aligned}$$

where $\mathcal{A}_i := \begin{bmatrix} A^0 & A^1 & \dots & A^i \end{bmatrix}$ and $\tilde{\mathcal{X}}_f$ is a robust invariant set for the system $x^+ = (A + BK)x$ for some stabilizing K.

We do **nominal MPC**, but with tighter constraints on the states and inputs.

We can be sure that if the nominal system satisfies the tighter constraints, then the uncertain system will satisfy the real constraints.

$\Rightarrow$ Downside is that $\mathcal{A}^i \mathbb{W}^i$ can be very large

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MPC as a Game

Two players: Controller vs Disturbance

$$x^+ = f(x, u) + w$$

1. Controller chooses his move u
2. Disturbance decides on his move w **after seeing the controller's move**

MPC as a Game

Two players: Controller vs Disturbance

$$x^+ = f(x, u) + w$$

1. Controller chooses his move u
2. Disturbance decides on his move w **after seeing the controller's move**

What are we assuming when making robust predictions?

1. Controller chooses a **sequence** of N moves in the future $\{u_0, \ldots, u_{N-1}\}$
2. Disturbance chooses N moves **knowing all N moves of the controller**

We are assuming that the controller will do the same thing in the future no matter what the disturbance does!

Can we do better?

Closed-Loop Predictions

- SWhat should the future prediction look like?

1. Controller decides his first move u_0
2. Disturbance chooses his first move w_0
3. Controller decides his second move $u_1(x_1)$ as a function of the first disturbance w_0 (recall $x_1 = Ax_0 + Bu_0 + w_0$)
4. Disturbance chooses his second move w_1 as a function of u_1
5. Controller decides his second move $u_2(x_2)$ as a function of the first two disturbances w_0, w_1
6. . . .

Closed-Loop Predictions

We want to optimize over a **sequence of functions** $\{u_0, \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$, where $\mu_i(x_i) : \mathbb{R}^n \to \mathbb{R}^m$ is called a **control policy**, and maps the state at time i to an input at time i.

Notes:

- This is the same as making μ a function of the disturbances to time i, since the state is a function of the disturbances up to that point
- The first input u_0 is a function of the current state, which is known. Therefore it is not a function, but a single value.

The problem: We can't optimize over arbitrary functions!

Closed-Loop Predictions

A solution: Assume some structure on the functions μ_i

Pre-stabilization $\mu_i(x) = Kx + v_i$

- Fixed K, such that $A + BK$ is stable
- Simple, often conservative

Linear feedback $\mu_i(x) = K_i x + v_i$

- Optimize over K_i and v_i
- Non-convex. Extremely difficult to solve...

Disturbance feedback $\mu_i(x) = \sum_{j=0}^{i-1} M_{ij} w_j + v_i$

- Optimize over M_{ij} and v_i
- Equivalent to linear feedback, but convex!
- Can be very effective, but computationally intense.

Tube-MPC $\mu_i(x) = v_i + K(x - \bar{x}_i)$

- Fixed K, such that $A + BK$ is stable
- Optimize over $\bar{x}_i$ and v_i
- Simple, and can be effective

We will cover tube-MPC in this lecture.

Robust MPC

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Tube MPC

$$x^+ = Ax + Bu + w \qquad (x, u) \in \mathcal{X} \times \mathcal{U} \qquad w \in \mathbb{W}$$

The idea: Seperate the available control authority into two parts

1. A portion that steers the noise-free system to the origin $z^+ = Az + Bv$
2. A portion that compensates for deviations from this system $e^+ = (A + BK)e + w$

We fix the linear feedback controller K offline, and optimize over the nominal trajectory $\{v_0, \ldots, v_{N-1}\}$, which results in a convex problem.

Tube MPC: System Decomposition

Define a 'nominal', noise-free system:

$$z_{i+1} = Az_i + Bv_i$$

Define a 'tracking' controller, to keep the real trajectory close to the nominal

$$u_i = K(x_i - z_i) + v_i$$

for some linear controller K, which stabilizes the nominal system.

Define the error $e_i = x_i - z_i$, which gives the error dynamics:

$$\begin{aligned} e_{i+1} &= x_{i+1} - z_{i+1} \\ &= Ax_i + Bu_i + w_i - Az_i - Bv_i \\ &= Ax_i + BK(x_i - z_i) + Bv_i + w_i - Az_i - Bv_i \\ &= (A + BK)(x_i - z_i) + w_i \\ &= (A + BK)e_i + w_i \end{aligned}$$

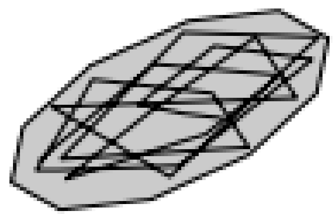
Tube MPC:Error Dynamics

Bound maximum error, or how far the 'real' trajectory is from the nominal

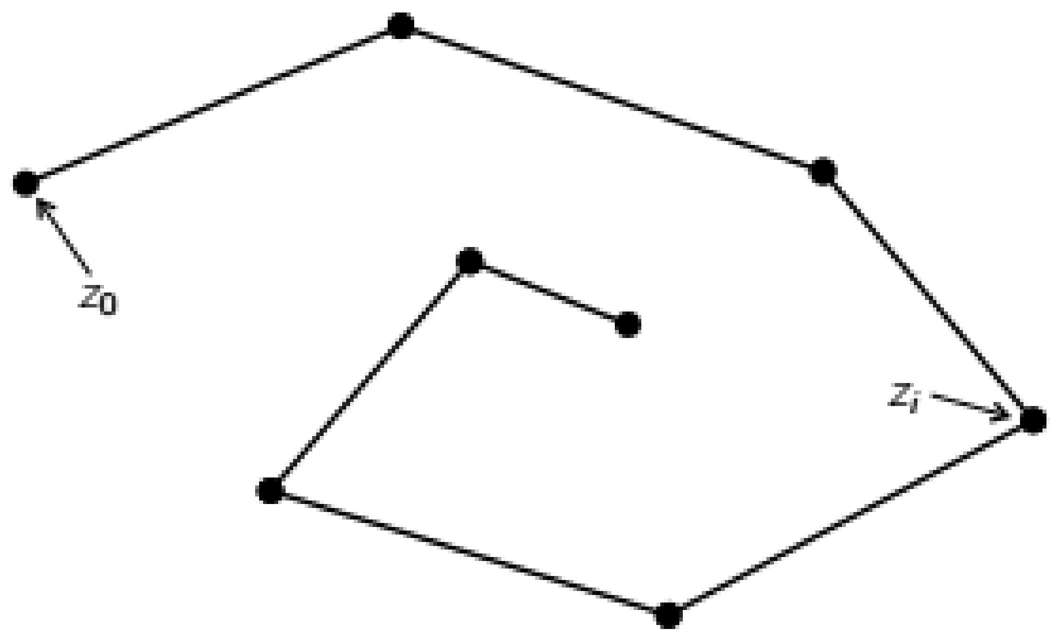
$$e_{i+1} = (A + BK)e_i + w_i \qquad\qquad w_i \in \mathbb{W}$$

Dynamics $A + BK$ are stable, and the set $\mathbb{W}$ is bounded, so there is some set $\mathcal{E}$ that e will stay inside for all time.

We want the smallest such set (the 'minimal invariant set')

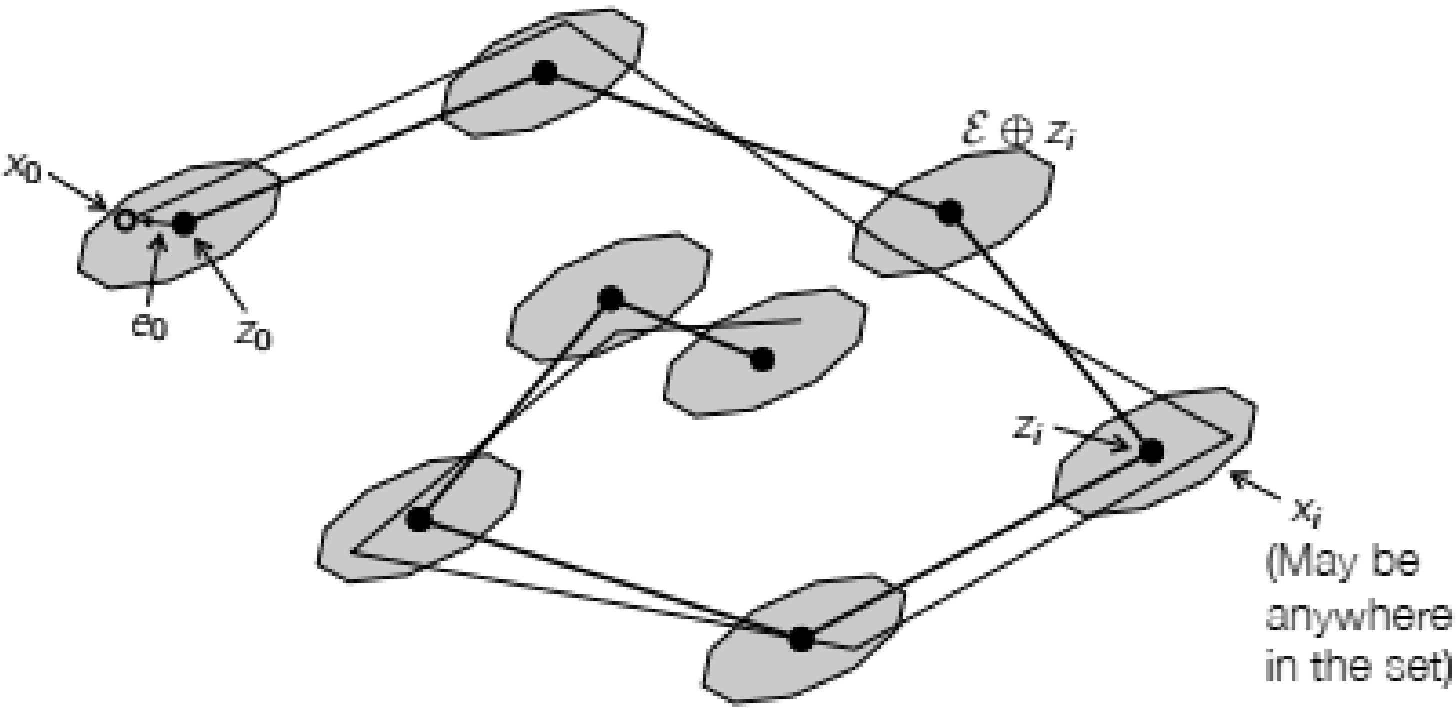


Tube MPC: The Idea



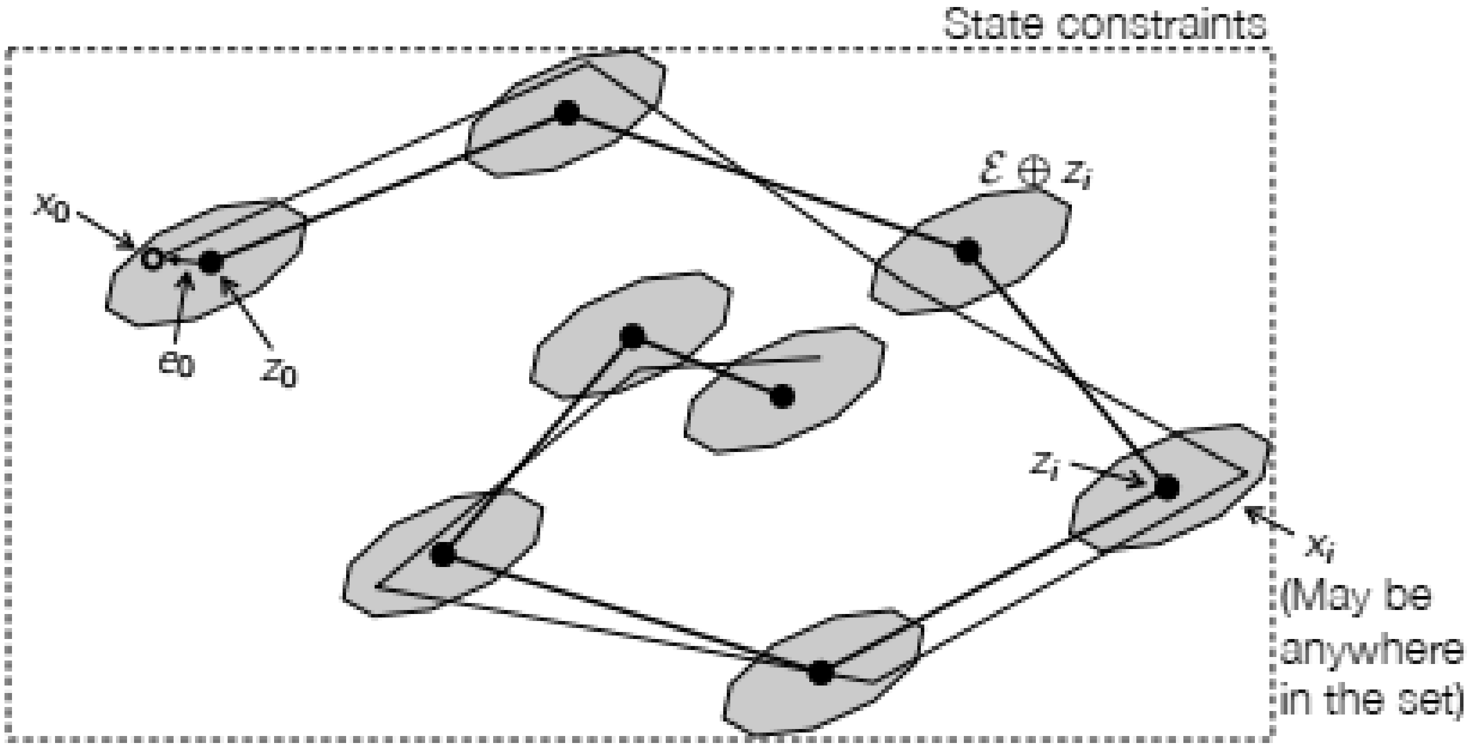
We want to ignore the noise and plan the **nominal trajectory**

Tube MPC: The Idea



We know that the real trajectory stays 'nearby' the nominal one: $x_i \in z_i \oplus \mathcal{E}$
because we plan to apply the controller $u_i = K(x_i - z_i) + v_i$ **in the future**
(we won't actually do this, but it's a valid sub-optimal plan)

Tube MPC: The Idea



We must ensure that all possible state trajectories satisfy the constraints

This is now equivalent to ensuring that $z_i \oplus \mathcal{E} \subset \mathcal{X}$

(Satisfying input constraints is now more complex - more later)

Tube MPC

What do we need to make this work?

- Compute the set $\mathcal{E}$ that the error will remain inside
- Modify constraints on nominal trajectory $\{z_i\}$ so that $z_i \oplus \mathcal{E} \subset \mathcal{X}$ and $v_i \in \mathcal{U} \ominus K\mathcal{E}$
- Formulate as convex optimization problem

. . . and then prove that

- Constraints are robustly satisfied
- The closed-loop system is robustly stable

Noisy System Trajectory

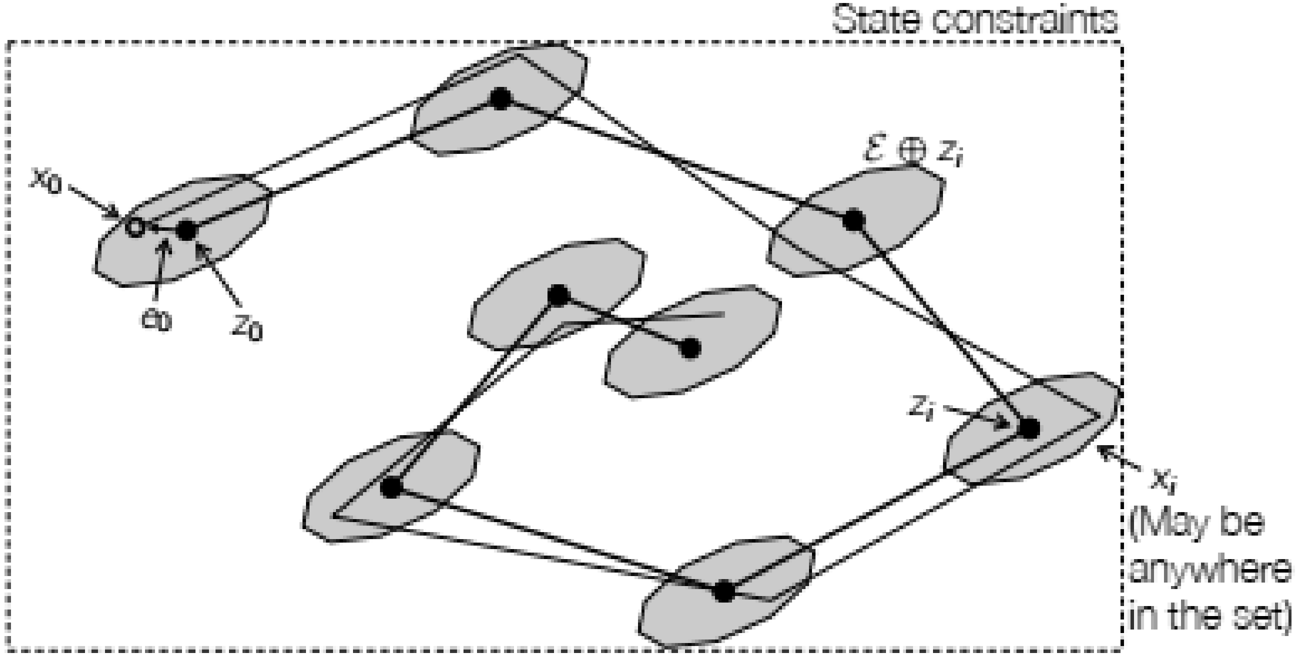
Given the nominal trajectory z_i, what can the noisy system trajectory do?

$$x_i = z_i + e_i$$

Don't know what error will be at time i, but it will be in the set $\mathcal{E}$

Therefore, x_i can only be up to $\mathcal{E}$ far from z_i

$$x_i \in z_i \oplus \mathcal{E} = \{z_i + e \,|\, e \in \mathcal{E}\}$$



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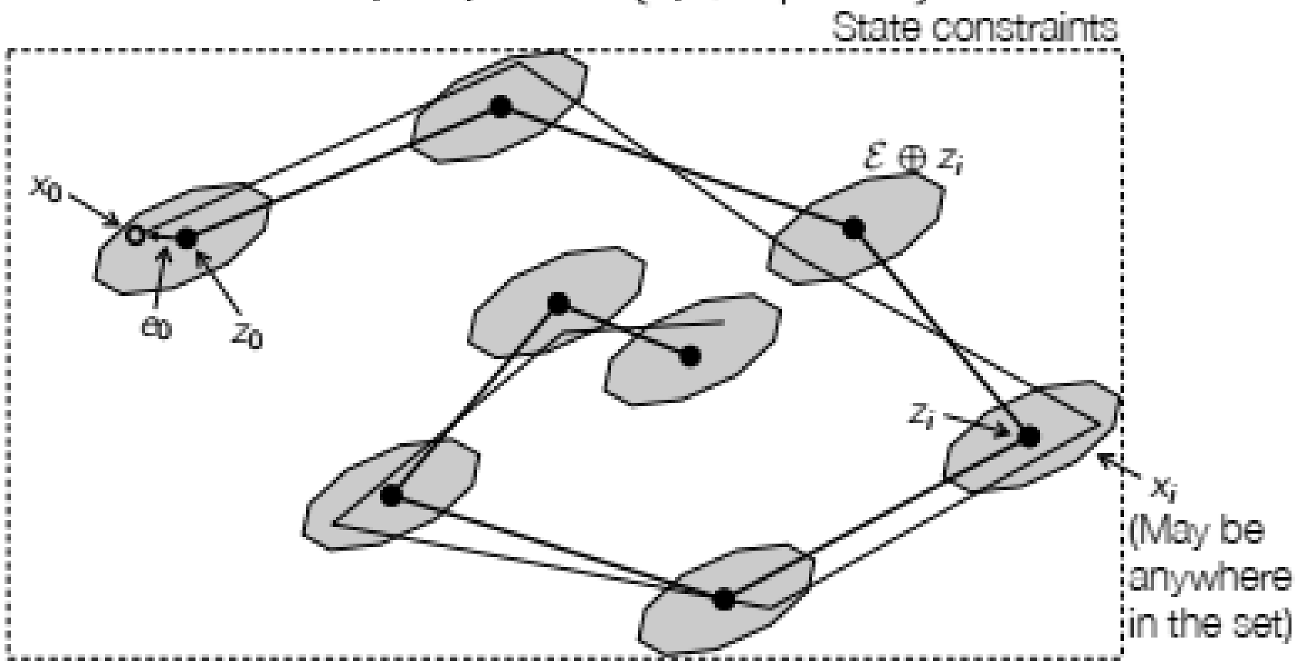


Constraint Tightening

Goal: $(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$ for all $\{w_0, \ldots, w_{i-1}\} \in \mathbb{W}^i$

We want to work with the nominal system $z^+ = Az + Bv$ but ensure that the noisy system $x^+ = Ax + Bu + w$ satisfies the constraints.

Sufficient condition:

$$z_i \oplus \mathcal{E} \subseteq \mathcal{X} \qquad \Leftarrow \qquad z_i \in \mathcal{X} \ominus \mathcal{E}$$

The set $\mathcal{E}$ is known offline - we can compute the constraints $\mathcal{X} \ominus \mathcal{E}$ offline!

A similar condition holds for the inputs:

$$u_i \in K\mathcal{E} \oplus v_i \subset \mathcal{U} \qquad \Leftarrow \qquad v_i \in \mathcal{U} \ominus K\mathcal{E}$$

Tube-MPC: Problem Formulation

Tube-MPC

Feasible set: $$\mathcal{Z}(x_0) := \left\{ \vec{z}, \vec{v} \;\middle|\; \begin{array}{ll} z_{i+1} = Az_i + Bv_i & i \in [0,\ N-1] \\ z_i \in \mathcal{X} \ominus \mathcal{E} & i \in [0,\ N-1] \\ v_i \in \mathcal{U} \ominus K\mathcal{E} & i \in [0,\ N-1] \\ z_N \in \mathcal{X}_f & \\ x_0 \in z_0 \oplus \mathcal{E} & \end{array} \right\}$$

Cost function: $$V(\vec{z}, \vec{v}) := \sum_{i=0}^{N-1} l(z_i, v_i) + V_f(z_N)$$

Optimization problem: $$(\vec{v}^*(x_0), \vec{z}^*(x_0)) = \underset{\vec{v}, \vec{z}}{\operatorname{argmin}} \{V(\vec{z}, \vec{v}) \,|\, (\vec{z}, \vec{v}) \in \mathcal{Z}(x_0)\}$$

Control law: $$\mu_{\text{tube}}(x) := K(x - z_0^*(x)) + v_0^*(x)$$

- Optimizing the nominal system, with tightened state an input constraints
- First tube center is optimization variable → has to be within $\mathcal{E}$ of x_0
- The cost is with respect to the tube centers
- The terminal set is with respect to the tightened constraints

Tube MPC - Example

System dynamics

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u + w \qquad \mathbb{W} := \{w \mid |w_1| \leq 0.01,\ |w_2| \leq 0.1\}$$

Constraints:

$$\mathcal{X} := \{x \mid \|x\|_\infty \leq 1\} \qquad \mathcal{U} := \{u \mid \|u\| \leq 1\}$$

Stage cost is:

$$l(z, v) := {z_i}^\top Q z_i + {v_i}^\top R v_i$$

where

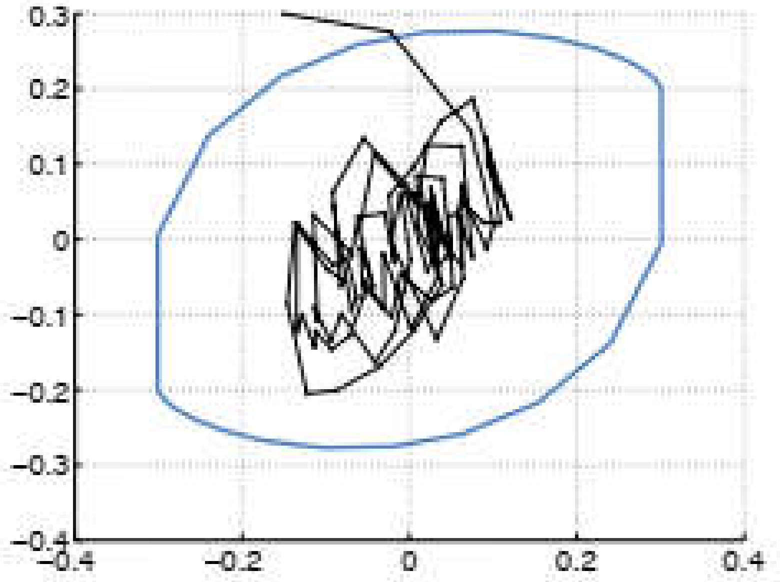
$$Q := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R := 10$$

Tube MPC - Example

1. Choose a stabilizing controller K so that $\|A + BK\| < 1$
2. Compute the minimal robust invariant set $\mathcal{E} = F_\infty$ for the system $x^+ = (A + BK)x + w,\ w \in \mathbb{W}$

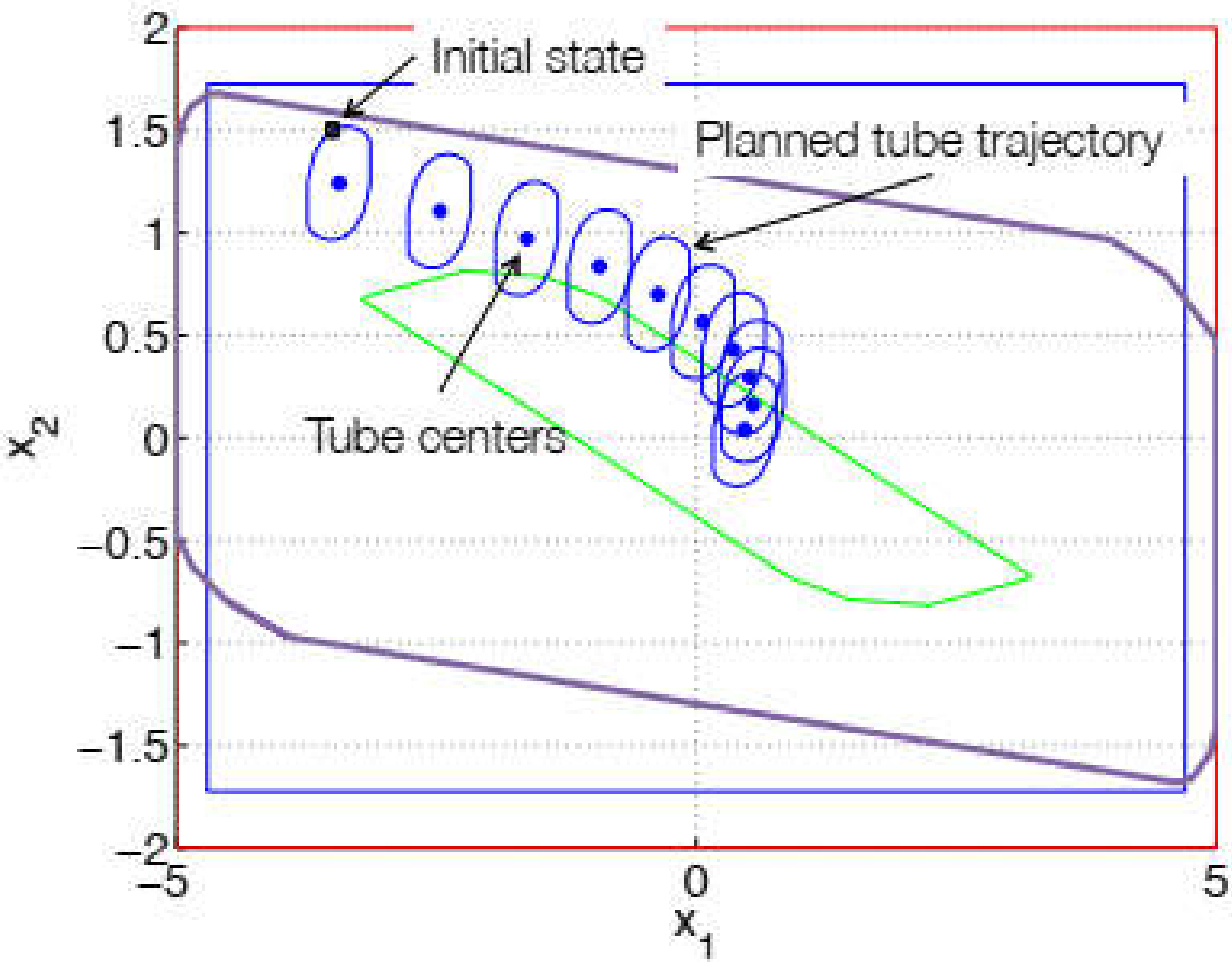
We take the LQR controller for $Q = I, R = 1$:

$$K := \begin{bmatrix} -0.5198 & -0.9400 \end{bmatrix}$$

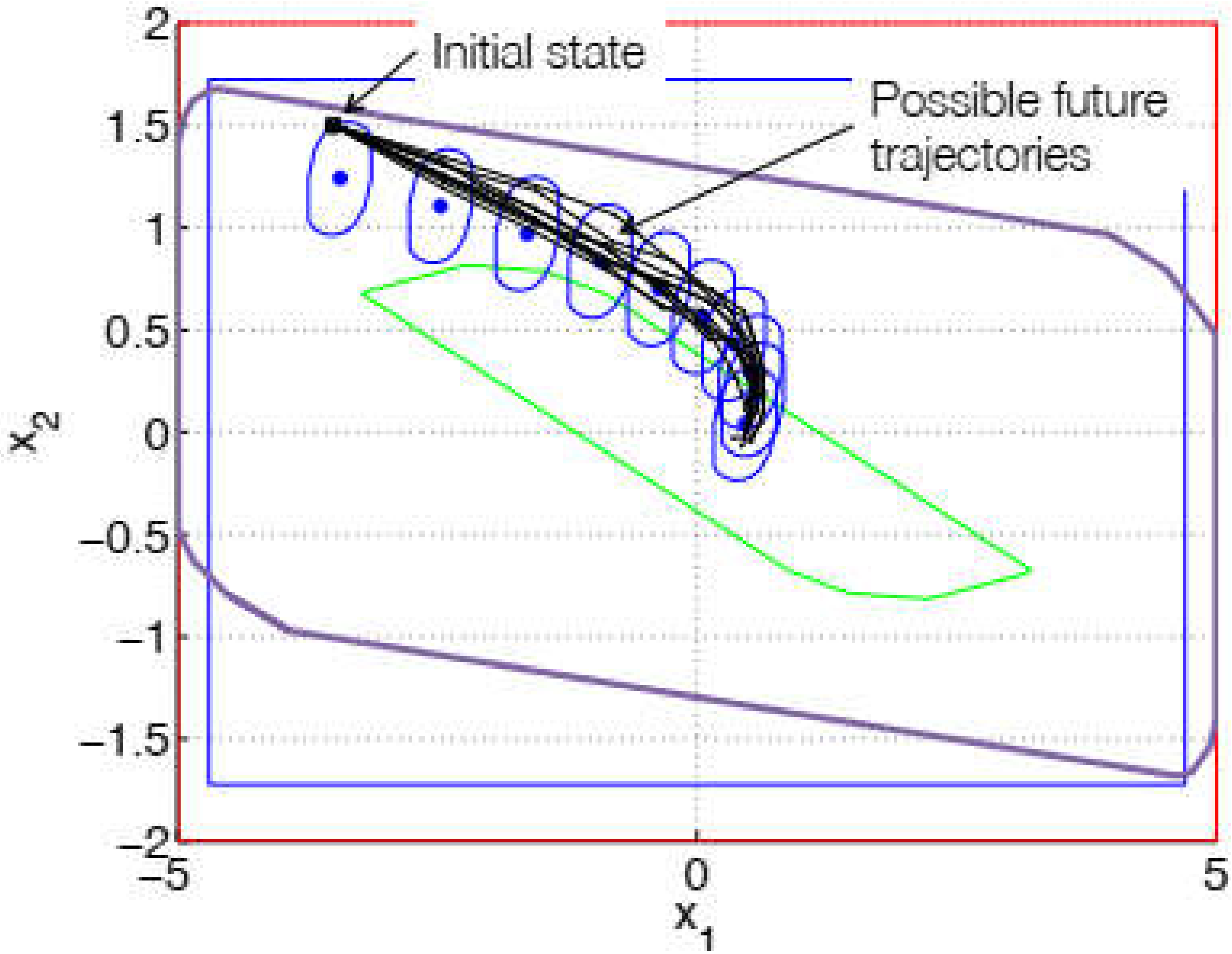


Evolution of the system $x^+ = (A + BK)x + w$ for $x_0 = \begin{bmatrix} -0.1 & 0.2 \end{bmatrix}^T$

Tube MPC - Example



Tube MPC - Example



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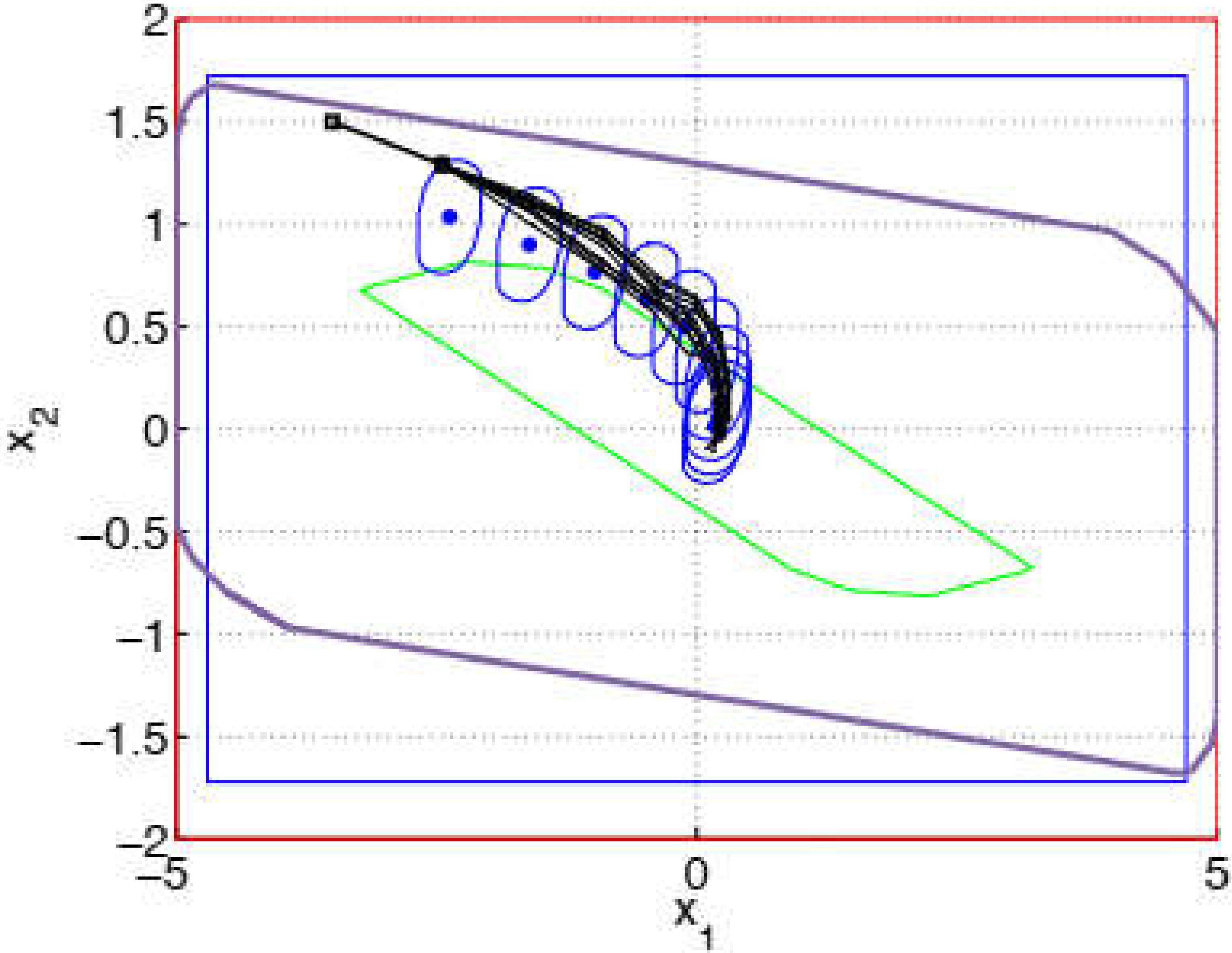

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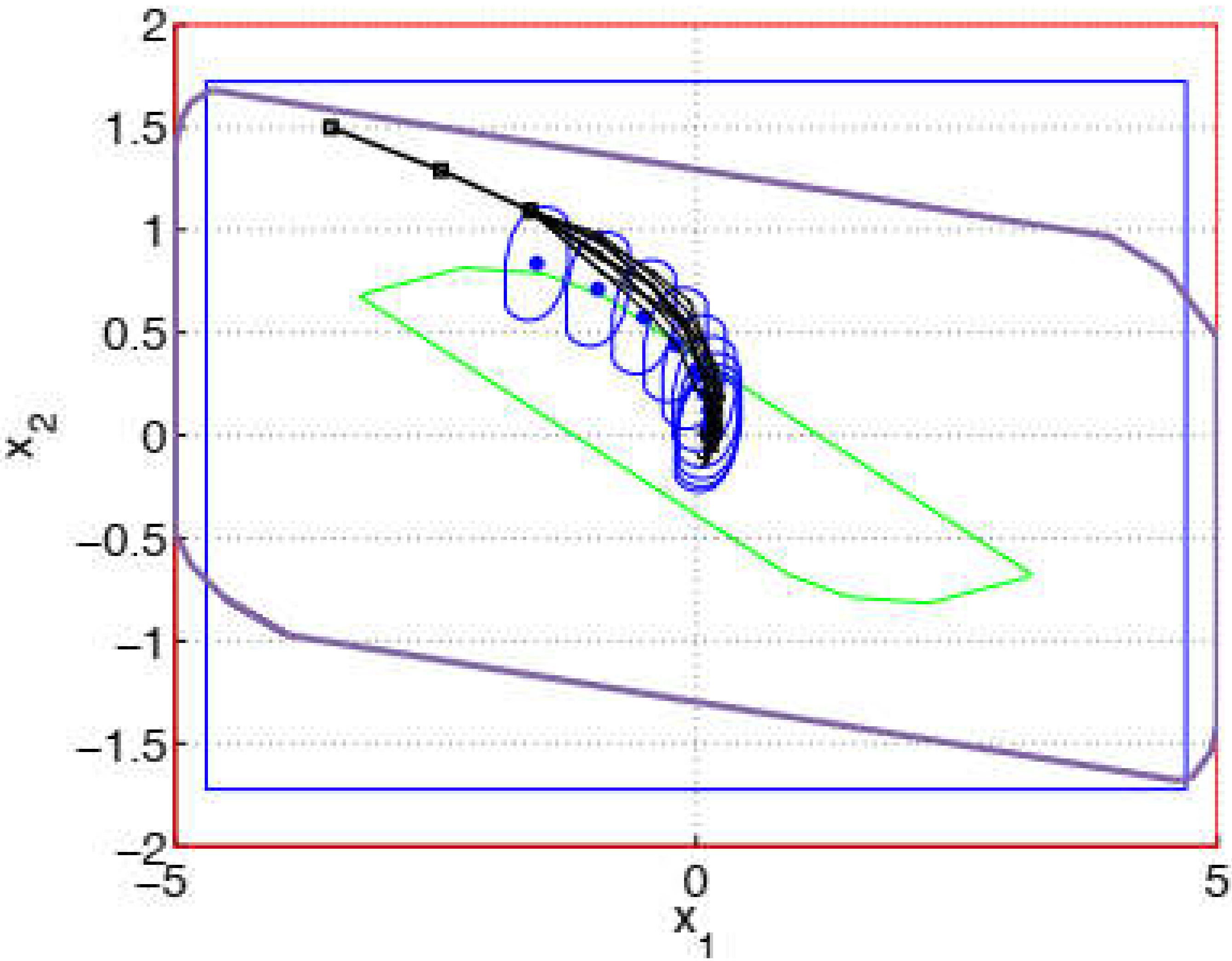

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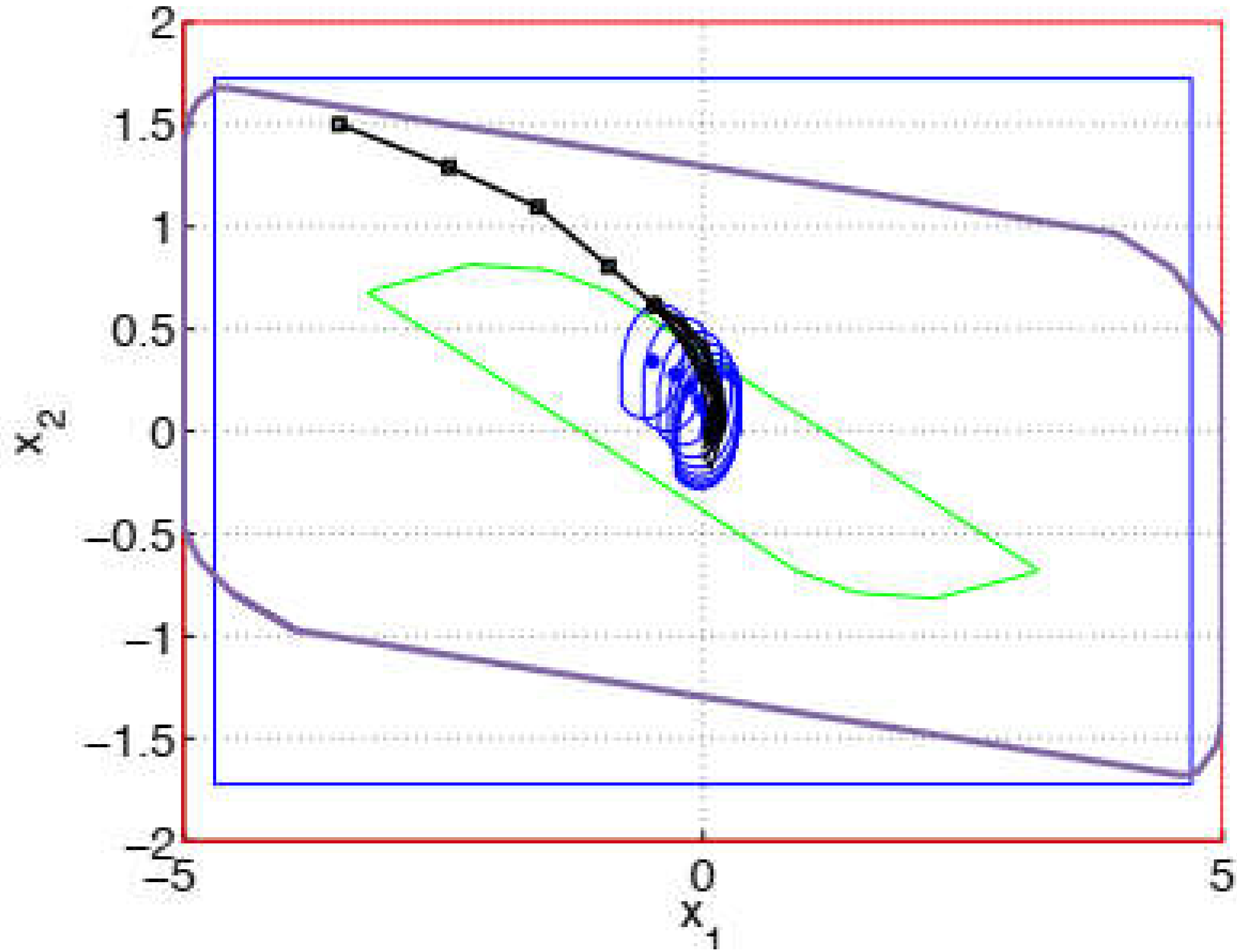
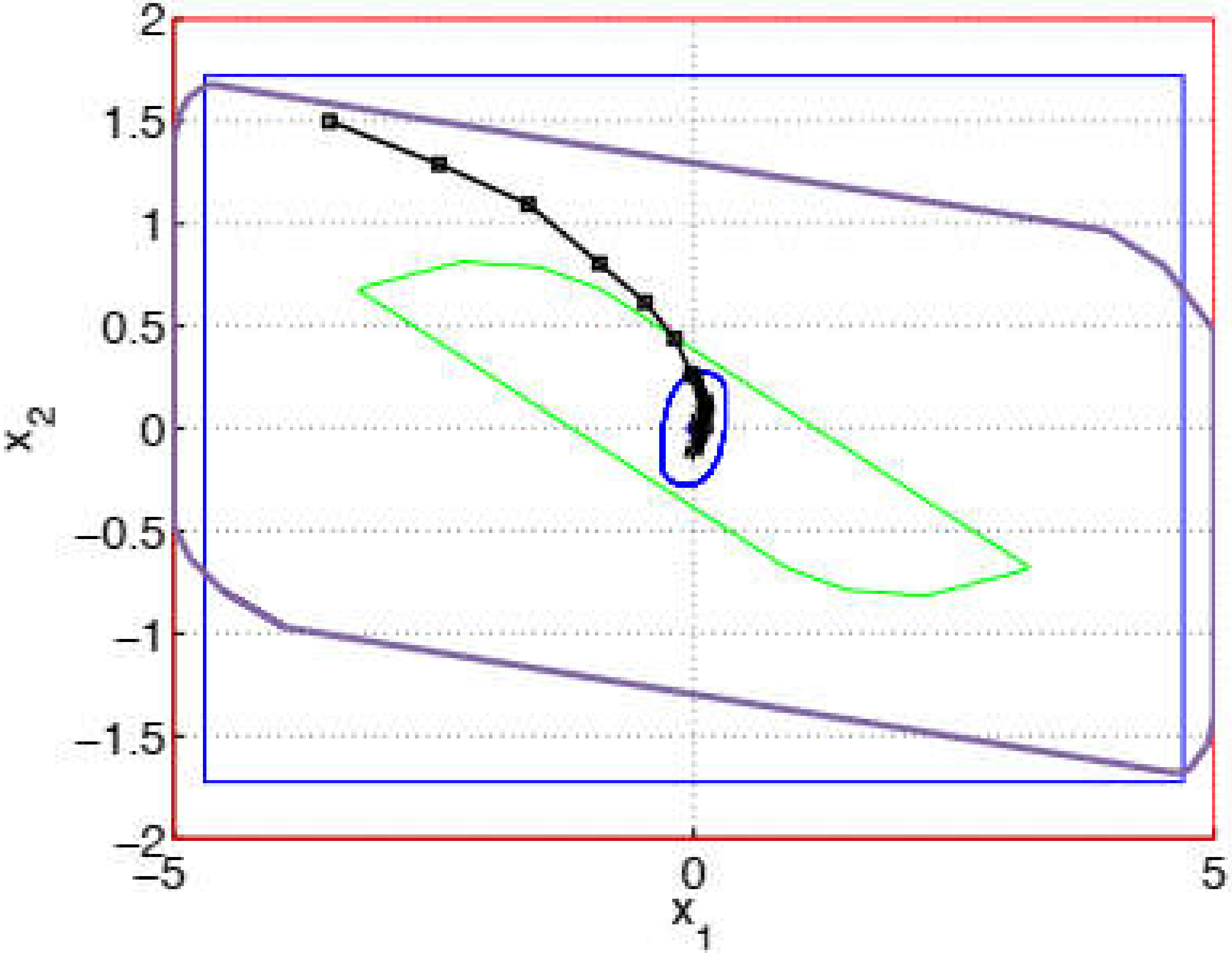


Tube MPC - Example



Tube MPC - Summary

Idea

- Split input into two parts: One to steer system (v), one to compensate for the noise (Ke)

$$u = Ke + v$$

- Optimize for the nominal trajectory, ensuring that any deviations stay within constraints

Benefits:

- Less conservative than open-loop robust MPC (we're now actively compensating for noise in the prediction)
- Works for unstable systems
- Optimization problem to solve is simple

Cons:

- Sub-optimal MPC (optimal is extremely difficult)
- Reduced feasible set when compared to nominal MPC
- We need to know what W is (this is usually not realistic)

Robust MPC for Uncertain Systems - Summary

Idea

- Compensate for noise in prediction to ensure all constraints will be met

Cons

- Complex (some schemes are simple to implement, like tubes, but complex to understand)
- Must know the largest noise W
- Often very conservative
- Feasible set may be small

Benefits

- Feasible set is invariant - we know exactly when the controller will work
- Easier to tune - knobs to tradeoff robustness against performance